Frobenius Problem

in Numerical Semigroups

Expository talk

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CONTENT

Numerical semigroups: definition

A numerical semigroup $\mathsf{S}(\mathbf{d}^m) = \langle d_1, \ldots, d_m \rangle$ is said to be generated by a minimal set of natural numbers $\mathbf{d}^m = \{d_1, \ldots, d_m\}$, $\gcd \mathbf{d}^m = 1$

$$\mathsf{S}(\mathbf{d}^m) = \left\{ s \in \mathbb{N} \cup \{0\} \mid s = \sum_{i=1}^{m} x_i d_i,\ x_i \in \mathbb{N} \cup \{0\} \right\},$$

if neither of its elements is linearly representable by the rest of them.

Notations

$d_1, \ldots, d_m$ - generators, $\pi_m = \prod_{i=1}^m d_i$, $\sigma_1 = \Sigma_{i=1}^m d_i$

$\mu(\mathbf{d}^m) = \min\{d_1, \ldots, d_m\}$ - a multiplicity,

m - an embedding dimension, $edim$,

$\mathbb{N} \setminus \mathsf{S}(\mathbf{d}^m)$ - a set of gaps of semigroup,

$\mathbf{F}(\mathbf{d}^m) = \max\{\mathbb{N} \setminus \mathsf{S}(\mathbf{d}^m)\}$ - the Frobenius number,

$\mathbf{g}(\mathbf{d}^m) = \#\{\mathbb{N} \setminus \mathsf{S}(\mathbf{d}^m)\}$ - a genus of semigroup,

$\mathbf{c}(\mathbf{d}^m) = 1 + \mathbf{F}(\mathbf{d}^m)$ - a conductor of semigroup,

$$\mathbf{c}(\mathbf{d}^m) \le 2\mathbf{g}(\mathbf{d}^m), \qquad \mathcal{Q}_m = F(\mathbf{d}^m) + \sigma_1$$

$\rho(\mathbf{d}^m) = 1 - \mathbf{g}(\mathbf{d}^m)/\mathbf{c}(\mathbf{d}^m)$ - a density of non-gaps

$Ap[\mathsf{S}(\mathbf{d}^m); a]$ - the Apéry set of $\mathsf{S}(\mathbf{d}^m)$ w.r.t.

$H(\mathbf{d}^m; z) = \sum_{s \in \mathsf{S}(\mathbf{d}^m)} z^s$, $z < 1$, - the Hilbert series

$\Phi(\mathbf{d}^m; z) = \sum_{s \notin \mathsf{S}(\mathbf{d}^m)} z^s$ - a generating function of gaps

$$H(\mathbf{d}^m; z) + \Phi(\mathbf{d}^m; z) = \frac{1}{1-z}, \quad z < 1$$

Frobenius problem – basic facts

According to oral (unpublished) reminiscences of I. Schur,
"... G. Frobenius in his lectures repeatedly raised the question of determining (or bounding) $\mathbf{F}(\mathbf{d}^m)$"

A. Brauer (1942): The Theorems in §3-5 result partly from discussions of Schur and the author. It was formerly intended to publish these results in a joint paper. I conform with Schur's wishes that the publishing be not longer postponed and that I publish the paper alone.

The paper was submitted for publication in 1940, less than two months before Schurs death, and was published two years later.

Curtis (1990): *There is no polynomial* $\Psi \in \mathbb{C}[X_1, \ldots, X_m, Y]$, $\Psi \not\equiv 0$, *such that*

$$\Psi(d_1, \ldots, d_m, F(\mathbf{d}^m)) = 0,$$

for each choice of $\mathbf{d}^m$, $m \geq 3$

Kannan (1992): *There exists a polynomial time algorithm for any fixed* m *but the time depends at least exponentially on* m

Ramírez-Alfonsín (1996): *The Frobenius problem is NP-hard under Turing-reductions*

Applications

Combinatorics, Number theory, Commutative algebra,
Algebraic Geometry, Probability, Computer science

Special numerical semigroups $\mathsf{S}(\mathbf{d}^m)$

Roberts (1956): Arithmetic sequence

$$\mathbf{d}^m = \{a, a+d, \dots, a+(m-1)d\}$$

$$F(\mathbf{d}^m) = a\left(\left\lfloor \frac{a-2}{m-1} \right\rfloor + 1\right) + (d-1)(a-1) - 1$$

Selmer (1997), Rödseth (1999): Almost arithmetic sequence

$$\mathbf{d}^m = \{a, ha+d, ha+2d \dots, ha+(m-1)d\}$$

$$F(\mathbf{d}^m) = ha\left\lfloor \frac{a-2}{m-1} \right\rfloor + a(h-1) + d(a-1)$$

$$\mathbf{d}^m = \{a, a+1, a+2, a+2^2 \dots, a+2^{m-2}\}$$

$$F(\mathbf{d}^m) = \frac{a(a+1)}{2^{m-2}} + \sum_{k=0}^{m-3} 2^k \left\lfloor \frac{a+2^k}{2^{m-2}} \right\rfloor + a(m-4) - 1$$

Ong, Ponomarenko (2008): Geometric sequence

$$\mathbf{d}^m = \{a^{m-1}, a^{m-2}b, a^{m-3}b^2, \dots, b^{m-1}\}, \quad b > a, \quad \gcd(a,b) = 1$$

$$F(\mathbf{d}^m) = b^{m-2}(ab-a-b) + \frac{(b-1)a^2(a^{m-2}-b^{m-2})}{a-b}$$

The Apéry set of $\mathsf{S}(\mathbf{d}^m)$ w.r.t. $a \in \mathsf{S}(\mathbf{d}^m),\ a \neq 0$

$$Ap[\mathsf{S}(\mathbf{d}^m); a] = \{s \in \mathsf{S}(\mathbf{d}^m) \mid s - a \notin \mathsf{S}(\mathbf{d}^m)\}$$

Schockley, Brauer (1962)

$$\#Ap[\mathsf{S}(\mathbf{d}^m); a] = a, \qquad F(\mathbf{d}^m) = \max Ap[\mathsf{S}(\mathbf{d}^m); a] - a$$

Example: $Ap[\langle 7, 12, 13\rangle, 7] = \{0, 12, 13, 24, 25, 36, 37\},\ F = 30$

Symmetric semigroups

A semigroup $\mathsf{S}(\mathbf{d}^m)$ is called *symmetric* if for any integer s the following condition holds: if $s \in \mathsf{S}(\mathbf{d}^m)$ then $F(\mathbf{d}^m) - s \notin \mathsf{S}(\mathbf{d}^m)$. Otherwise $\mathsf{S}(\mathbf{d}^m)$ is called nonsymmetric.

$$\mathbf{c}(\mathbf{d}^m) = 2\mathbf{g}(\mathbf{d}^m), \qquad \rho(\mathbf{d}^m) = \frac{1}{2}$$

If $\mu(\mathbf{d}^m) = m$ then semigroup $\mathsf{S}(\mathbf{d}^m)$ is never symmetric

Examples

$$\mathsf{S}_1 = \langle 4, 7\rangle = \{0, 4, 7, 8, 11, 12, 14, 15, 16, 18, \longmapsto\},$$
$$\mathsf{G}_1 = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}, \quad \mathbf{F}_1 = 17, \quad \mathbf{c}_1 = 18, \quad \mathbf{g}_1 = 9,$$
$$\mathsf{S}_2 = \langle 4, 5, 6, 7\rangle = \{0, 4, \longmapsto\}, \qquad \mathsf{G}_2 = \{1, 2, 3\},$$
$$\mathbf{F}_2 = 3, \qquad \mathbf{c}_2 = 4, \qquad \mathbf{g}_2 = 3, \qquad \rho = 1/4$$

Watanabe (1973): *Let $H_1 = \langle d_1, \ldots, d_m\rangle$ be a semigroup, and $a, b \in \mathbb{Z}_+$ such that:*

- $a \in H_1 \setminus \{d_1, \ldots, d_m\}$,
- $\gcd(a, b) = 1$

Denote by $H = \langle a, bH_1\rangle$ a semigroup $H = \langle a, bd_1, \ldots, bd_k\rangle$, then H is symmetric if and only if H_1 is symmetric.

Example

$$\mathsf{H} = \langle 20, 21, 24, 27, 39\rangle, \quad \mathsf{H}_1 = \langle 7, 8, 9, 13\rangle\rangle, \quad 20, \quad b = 3$$

Johnson (1960):

$$F(H) = bF(H_1) + (b-1)a$$

Telescopic semigroups

$\mathsf{S}(\mathbf{d}^m)$ is said to be telescopic iff for all $2 \le k \le m$,

$$\frac{d_k}{g_k} \in \mathsf{S}_{k-1}, \quad \mathsf{S}_k = \mathsf{S}\left(\left\{\frac{d_1}{g_k}, \frac{d_2}{g_k}, \ldots, \frac{d_k}{g_k}\right\}\right), \quad g_k = \gcd(d_1, \ldots, d_k) > 1$$

Example $\mathsf{H} = \langle 11\langle 5\langle 2,3\rangle, 7\rangle, 17\rangle = \langle 17, 77, 110, 165\rangle$

Delorme (1976): *Let $H_1 = \langle d_1, \ldots, d_m\rangle$ and $H_2 = \langle c_1, \ldots, c_k\rangle$ be semigroups, and $a, b \in \mathbb{Z}_+$ such that:*

- $a \in H_1 \setminus \{d_1, \ldots, d_m\}$, • $b \in H_2 \setminus \{c_1, \ldots, c_k\}$, • $\gcd(a, b) = 1$

Denote by $H = \langle bH_1, aH_2\rangle = \langle bd_1, \ldots, bd_m, ac_1, \ldots, ac_k\rangle$, then H is symmetric if and only if H_1 and H_2 are symmetric.

Example

$\mathsf{H} = \langle 52, 54, 65, 63\rangle, \quad \mathsf{H}_1 = \langle 4, 5\rangle, \quad \mathsf{H}_2 = \langle 6, 7\rangle, \quad a = 9, \quad b = 13$

Semigroups $\mathsf{S}(\mathbf{d}^2)$

The semigroup $\mathsf{S}(\mathbf{d}^2)$ is always symmetric

Sylvester (1884), Rödseth (1994):

$$F\left(\mathbf{d}^2\right) = d_1 d_2 - d_1 - d_2, \qquad g\left(\mathbf{d}^2\right) = \frac{(d_1 - 1)(d_2 - 1)}{2},$$

$$H\left(\mathbf{d}^2; z\right) = \frac{1 - z^{d_1 d_2}}{(1 - z^{d_1})(1 - z^{d_2})}, \qquad g_n\left(\mathbf{d}^2\right) = \sum_{s \text{ gaps}} s^n,$$

$$g_n\left(\mathbf{d}^2\right) = K_n \sum_{k=0}^{n+1} \sum_{l=0}^{n+1-k} \binom{n+2}{k} \binom{n+2-k}{l} B_k B_l d_1^{n+1-k} d_2^{n+1-l} - B_{n+1}/(n+1), \qquad K_n = ((n+1)(n+2))^{-1}$$

Pseudo-symmetric semigroups

A numerical semigroup $\mathsf{S}(\mathbf{d}^m)$ is said to be pseudo-symmetric if

$$\mathbf{c}(\mathbf{d}^m) = 2\mathbf{g}(\mathbf{d}^m) - 1$$

Arf semigroups

A numerical semigroup $\mathsf{S}(\mathbf{d}^m)$ is called Arf semigroup if

$$s_i + s_j - s_k \in \mathsf{S}(\mathbf{d}^m) \quad \text{for every} \quad i,j,k \in \mathbb{N} \quad \text{and} \quad i \geq j \geq k$$

Munuera, Torres, Villanueva (2009)

Arf semigroups are sparse. Let $\mathsf{S}(\mathbf{d}^m)$ *is Arf semigroup and* $s_i, s_{i+1} < \mathbf{c}(\mathbf{d}^m)$ *are two non-gaps. Then* $s_{i+1} - s_i \geq 2$

The numerical semigroup is Arf iff for every two positive integers i, j *with* $i > j$ *it holds* $2s_i - s_j \in \mathsf{S}(\mathbf{d}^m)$

Hyperelliptic semigroups

A semigroup $\mathsf{S}(\mathbf{d}^2)$ is called hyperelliptic if it is generated by 2 and an odd integer $2k+1$. It is of the form

$$\mathsf{S}(\mathbf{d}^m) = \{0, 2, 4, \ldots, 2k-2, 2k, \longmapsto\}$$

Campillo, Farran, Munuera, (2000)

The only Arf symmetric semigroups are hyperelliptic semigroups

Bras-Amores (2016)

The only Arf pseudo-symmetric semigroups are $\{0, 3, \longmapsto\}$ *and* $\{0, 3, 5, \longmapsto\}$

Semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$ and matrix of minimal relations

Let a nonsymmetric semigroup $S_3 = \langle d_1, d_2, d_3 \rangle$ be given by matrix of minimal relations, $\mathbb{A}_3$, $a_{ij} \in \mathbb{Z}_+$,

$$\mathbb{A}_3 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbb{A}_3 = \begin{pmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix},$$

$$a_{11} = \min\left\{v_{11} \mid v_{11} \geq 2,\ v_{11}d_1 = v_{12}d_2 + v_{13}d_3,\ v_{12}, v_{13} \in \mathbb{N} \cup \{0\}\right\}$$
$$a_{22} = \min\left\{v_{22} \mid v_{22} \geq 2,\ v_{22}d_2 = v_{21}d_1 + v_{23}d_3,\ v_{21}, v_{23} \in \mathbb{N} \cup \{0\}\right\}$$
$$a_{33} = \min\left\{v_{33} \mid v_{33} \geq 2,\ v_{33}d_3 = v_{31}d_1 + v_{32}d_2,\ v_{31}, v_{32} \in \mathbb{N} \cup \{0\}\right\}$$

All matrix elements a_{ij} are non-negative integers such that

$$a_{11} = a_{21} + a_{31}, \quad a_{22} = a_{12} + a_{32}, \quad a_{33} = a_{13} + a_{23},$$
$$d_1 = a_{22}a_{33} - a_{23}a_{32}, \quad d_2 = a_{33}a_{11} - a_{31}a_{13},$$
$$d_3 = a_{11}a_{22} - a_{12}a_{21}.$$

Hilbert series $H(z, S_3)$, Frobenius number $F(S_3)$, genus $g(S_3)$

$$H(z, S_3) = \frac{1 - z^{e_1} - z^{e_2} - z^{e_3} + z^{t_1} + z^{t_2}}{\left(1 - z^{d_1}\right)\left(1 - z^{d_2}\right)\left(1 - z^{d_3}\right)},$$

$$e_i = a_{ii}d_i, \qquad t_1 = \delta_0 + \delta_1, \qquad t_2 = \delta_0 + \delta_2,$$
$$\delta_0 = a_{11}a_{22}a_{33}, \qquad \delta_1 = a_{12}a_{23}a_{31}, \qquad \delta_2 = a_{13}a_{32}a_{21},$$
$$e_1 + e_2 + e_3 = t_1 + t_2, \qquad F_1 = t_1 - \sigma_1, \qquad F_2 = t_2 - \sigma_1,$$
$$F(S_3) = \max\left\{F_1, F_2\right\}, \qquad 2g(S_3) = 1 + \delta_0 + \delta_1 + \delta_2 - \sigma_1$$

LF (2004): *Let a nonsymmetric numerical semigroup* S_3 *be given. Then the matrix* A_3 *of minimal relations has a unique Rep.*

Rosales, Gárcia-Sánchez; LF (2004):

$$\mathbf{a} = \{a_{11}, a_{22}, a_{33}\}, \qquad \langle \mathbf{a}, \mathbf{d}^3 \rangle = a_{11}d_1 + a_{22}d_2 + a_{33}d_3,$$
$$\mathbf{g}\left(\mathbf{d}^m\right) = \frac{1}{2}\left[1 + \langle \mathbf{a}, \mathbf{d}^3 \rangle - \sigma_1 - \delta_0\right],$$
$$\mathbf{F}\left(\mathbf{d}^3\right) = \frac{1}{2}\left[\langle \mathbf{a}, \mathbf{d}^3 \rangle + J\left(\mathbf{d}^3\right)\right] - \sigma_1,$$
$$J^2\left(\mathbf{d}^3\right) = \langle \mathbf{a}, \mathbf{d}^3 \rangle^2 - 4\sum_{i>j}^{3} a_{ii}a_{jj}d_id_j + 4\pi_3$$

Lower and Upper Bounds

Rödseth (1990), Davison (1994):

$$\mathcal{Q}_3 > \sqrt{3}\sqrt{\pi_3},$$

LF (2004):

$$\mathcal{Q}_3 > \sqrt{3}\sqrt{\pi_3 + 1}$$

Beck, Einstein, Zack (BEZ) conjecture (2003):
Based on the thousands randomly chosen admissible triples $(d1, d2, d3)$ *satisfying the condition* $\sqrt{\pi_3} < 2\ 10^4$

$$\mathcal{Q}_3 < C\pi_3^{\nu}, \qquad \nu < 2/3$$

LF (2004): $\mathcal{Q}_3$ *cannot be bounded by any* $\mathcal{R}_{BEZ}$ *given by*

$$\mathcal{Q}_{BEZ} = C\pi_3^{\nu} \quad \textit{for any} \quad \nu < 2/3 \quad \textit{and} \quad C > 0$$

Short proof:

Consider a semigroup $S_3=\langle 2l+1,2l+3,4l+3\rangle$ *with matrix* $\mathbb{A}_3$ *of minimal relations,*

$$\mathbb{A}_3=\left(\begin{array}{ccc} l+3 & -l & -1\\ -l & l+1 & -1\\ -3 & -1 & 2\end{array}\right),\quad \gcd(2l+1,2l+3,4l+3)=1,$$

$$F(S_3)=2l^2+3l-1,\qquad \mathcal{Q}_3=2l^2+11l+6$$

Denote by $\delta_{C;\nu}(l)$ the ratio

$$\delta_{C;\nu}(l)=\frac{\mathcal{Q}_{BEZ}}{\mathcal{Q}_3}$$

and note that the leading asymptotic term of $\delta_{C;\nu}(l)$ when $l\gg 1$ reads,

$$\delta_{C;\nu}(l)\simeq C\;2^{4\nu-1}l^{3\nu-2}$$

and leads to refuting the BEZ conjecture:

$$\delta_{C;\nu}(l)<1\qquad \text{if}\qquad l>l_{cr}\quad \text{where}\quad \lg_2 l_{cr}=\frac{4\nu-1}{2-3\nu}+\frac{\lg_2 C}{2-3\nu}$$

Example: $\nu=5/8,\quad C=1,\quad l_{cr}=4096,\quad l=5000$

$d_1=10001=73\cdot 137,\quad d_2=10003=7\cdot 1429,\quad d_3=20003=83\cdot 241,$

$$\mathcal{Q}_3=50014999,\qquad \mathcal{Q}_{BEZ}=48745746.76$$

Example: $\nu=5/8,\quad C=1.35,\quad l_{cr}=45189,\quad l=50000$

$$d_1=100001=11\cdot 9091,\quad d_2=100003,\quad d_3=200003,$$

$$\mathcal{Q}_3=5000149999,\qquad \mathcal{Q}_{BEZ}=3656883908.53$$

Semigroup series

LF (2004), LF, Rubinstein (2007)

$$g_n\left(\mathbf{d}^3\right) = \sum_{s\ gaps} s^n, \qquad n \geq 0,$$

$$g_0\left(\mathbf{d}^3\right) = \frac{1}{2}\left[1 - \sigma_1 - \delta_0 + \langle \mathbf{a}, \mathbf{d}^3 \rangle\right],$$

$$g_1\left(\mathbf{d}^3\right) = \frac{1}{12}\left[-1 + \pi_3 + \sum_{i=1}^{3} A_i d_i^2 + \sum_{i>j}^{3} B_{ij} d_i d_j - \delta_0 \sum_{i=1}^{3} C_i d_i\right],$$

$$A_i = (a_{ii}-1)(2a_{ii}-1), \qquad C_i = 2a_{ii} - 3,$$
$$B_{ij} = 3(a_{ii}-1)(a_{jj}-1) - a_{ii}a_{jj}$$

The other formulas for $n \geq 2$ are too long to be presented here.

LF, Komatsu (2017)

$$g_n\left(\mathbf{d}^2\right) = \sum_{s \in \mathbb{N} \setminus S_2} s^{-n}, \quad n \geq 1, \qquad S_2 = \langle d_1, d_2 \rangle,$$

$$g_{-n}(S_2) = \left(1 - \frac{1}{d_2^n}\right)\zeta(n) - \frac{1}{d_2^n}\sum_{k=1}^{d_2-1} \zeta\left(n, k\frac{d_1}{d_2}\right),$$

where $\zeta(n,q)$ and $\zeta(n)$ stand for the Hurwitz and Riemann zeta functions,

$$\zeta(n,q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^n}, \qquad \zeta(n,1) = \zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

Special Numerical Semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$

Kraft (1985): Pythagorean semigroup: $d_1^2 + d_2^2 = d_3^2$

$\langle k_1^2 - k_2^2,\ 2k_1k_2,\ k_1^2 + k_2^2 \rangle$

$$F\left(\mathbf{d}^3\right) = k_1\left(k_1^2 - k_2^2 + 2(k_1k_2 - k_1 - k_2)\right)$$

Marin, Ramirez Alfonsin, Revuelta (2007): Fibonacci semigroup

$\langle \{F_i, F_{i+1}, F_{i+k}\rangle, \quad i,k \geq 3, \qquad r = \lfloor (F_i - 1)/F_k \rfloor$

$$F\left(\mathbf{d}^3\right) = \begin{cases} (F_i - 1)F_{i+2} - F_i(rF_{k-2} + 1) & if \quad \mathcal{C}_1, \\ (rF_k - 1)F_{i+2} - F_i((r-1)F_{k-2} + 1) & if \quad \mathcal{C}_2, \end{cases}$$

$\mathcal{C}_1: \quad \text{if} \quad r = 0 \quad \text{or} \quad r \geq 1 \quad \text{and} \quad F_{k-2}F_i < (F_i - rF_k)F_{i+2}$

$\mathcal{C}_2: \quad \text{otherwise}$

LF (2007): Lucas symmetric semigroup

$\langle L_k, L_m, L_n \rangle, \quad k,m,n \geq 2$

Lepilov, O'Rourke, Swanson (2015)

$\langle (n-1)^k, n^k, (n+1)^2 \rangle, \quad n \neq 3,4,5 \quad \textit{congruence modulo} \quad T_2 = 4$

$\langle (n-1)^3, n^3, (n+1)^3 \rangle, \quad n \neq 3 \quad \text{congruence modulo} \quad T_3 = 18$

LF (2016)

$\langle (n-1)^4, n^4, (n+1)^4 \rangle, \quad n \neq 3,5,7 \quad \text{congruence modulo} \quad T_4 = 40$

Matrix $\mathbb{A}_3$ for semigroups $\langle (n-1)^2, n^2, (n+1)^2 \rangle$:

$$\left(\begin{array}{ccc} 16m+1 & -8(2m-1) & -1 \\ -(15m+1) & 16m-7 & -(m-1) \\ -m & -1 & m \end{array}\right), \qquad \mathbf{n=4m}$$

$$\begin{array}{rcl} G & = & 4m(34m^2-21m+2), \\ F & = & 20, \quad m=1, \\ F & = & 272m^3-168m^2+m-2, \;\; m \geq 2. \end{array}$$

$$\left(\begin{array}{ccc} 7m+2 & -4(m-1) & -(3m+1) \\ -(3m+1) & 4m & -m \\ -(4m+1) & -4 & 4m+1 \end{array}\right), \qquad \mathbf{n=4m+1}$$

$$\begin{array}{rcl} G & = & 2m(32m^2+9m+1), \\ F & = & 112m^3+48m^2+8m-1, \;\; m \leq 3, \\ F & = & 128m^3-20m-5, \;\; m \geq 4. \end{array}$$

$$\left(\begin{array}{ccc} 9m+5 & -(8m-1) & -(m+1) \\ -(7m+4) & 8m+1 & -m \\ -(2m+1) & -2 & 2m+1 \end{array}\right), \qquad \mathbf{n=4m+2}$$

$$\begin{array}{rcl} G & = & m(80m^2+71m+16), \\ F & = & 312, \quad m=1, \\ F & = & 160m^3+128m^2+10m-9, \;\; m \geq 2. \end{array}$$

$$\left(\begin{array}{ccc} 5m+4 & -4m & -(m+1) \\ -(m+1) & 4(m+1) & -(3m+2) \\ -(4m+3) & -4 & 4m+3 \end{array}\right), \qquad \mathbf{n=4m+3}$$

$$\begin{array}{rcl} G & = & 2(32Nm^3+57m^2+33m+6), \\ F & = & 128m^3+2242+124m+19, \;\; m \geq 1. \end{array}$$

Semigroup Rings

Let $R = \mathsf{k}[X_1, \ldots, X_m]$ be a polynomial ring over a field k of characteristic 0 and

$$\psi_0 \;:\quad R \longmapsto \mathsf{k}\left[z^{d_1}, \ldots, z^{d_m}\right]$$

be the projection induced by $X_i = z^{d_i}$.

A semigroup ring $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^m\right)\right] = \mathsf{k}\left[z^{d_1}, \ldots, z^{d_m}\right]$ is a graded 1-dim local subring of R and has a presentation

$$\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^m\right)\right] = R/\mathsf{M}_1, \qquad \mathsf{M}_1 = \ker(\psi_0)$$

and is associated to affine monomial curve with parametrization

$$X_1 = z^{d_1}, \quad X_2 = z^{d_2}, \quad \ldots, \quad X_m = z^{d_m}$$

A graded R-module M_1 is minimally generated by a finite number β_1 of binomial generators P_k, $1 \leq k \leq \beta_1$,

$$P_k(X_1, \ldots, X_m) = X_1^{h_1^k} \cdot \ldots \cdot X_m^{h_m^k} - X_1^{q_1^k} \cdot \ldots \cdot X_m^{q_m^k},$$

$h_j^k, q_j^k \in \mathbb{Z}_{\geq}$, such that $\psi_0(P_k) = 0$. Associated arithmetic relations

$$\sum_{i=1}^{m} h_i^k d_i = \sum_{j=1}^{m} q_j^k d_j, \qquad C_{1,k} = \sum_{i=1}^{m} h_i^k d_i,$$

are called syzygies of the 1st kind with degrees $C_{1,k}$ and M_1 is called the 1st syzygy module of $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^m\right)\right]$, or ideal.

For $n \geq 2$ it may happen that any set of $P_k \in \mathsf{M}_1$ has their own relations. It give rise to existence of another free module R_1, such that

$$R_1 \overset{\psi_1}{\longmapsto} R \overset{\psi_0}{\longmapsto} \mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^m\right)\right], \qquad \mathrm{Im}(\psi_1) = \mathsf{M}_1 = \ker(\psi_0)$$

Going to the higher syzygies we arrive at the Hilbert Syzygy Theorem

Hilbert's Syzygy Theorem: *Let Let* $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^m\right)\right]$ *be a finitely generated module over the polynomial ring* $R = \mathsf{k}[X_1, \ldots, X_m]$. *Then there is a minimal free resolution*

$$0 \longmapsto R_n \overset{\psi_n}{\longmapsto} \ldots \overset{\psi_2}{\longmapsto} R_1 \overset{\psi_1}{\longmapsto} R \overset{\psi_0}{\longmapsto} \mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^m\right)\right] \longmapsto 0,$$
$$\mathrm{Im}(\psi_k) = \mathsf{M}_k = \ker(\psi_{k-1}), \quad 1 \le k \le n$$

where $R_1, \ldots, R_n$ *are finitely generated free* R*-modules and* $n < m$.

If $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^m\right)\right]$ is a graded 1-dim local subring of R then $n = m - 1$

An integer $\beta_k = \dim \mathsf{M}_k$ is called the k-th Betti number

$$1 - \beta_1 + \beta_2 - \ldots + (-1)^{m-1}\beta_{m-1} = 0, \quad \beta_1 \ge m - 1$$

The Hilbert series

$$H\left(\mathbf{d}^m; z\right) = \frac{Q\left(\mathbf{d}^m; z\right)}{\prod^m \left(1 - z^{d_i}\right)},$$

$$Q\left(\mathbf{d}^m; z\right) = 1 - Q_1\left(\mathbf{d}^m; z\right) + \cdots + (-1)^{m-1} Q_{m-1}\left(\mathbf{d}^m; z\right),$$

$$Q_i\left(\mathbf{d}^m; z\right) = \sum_{j=1}^{\beta_i(\mathbf{d}^m)} z^{C_{j,i}}, \quad \deg Q_i\left(\mathbf{d}^m; z\right) < \deg Q_{i+1}\left(\mathbf{d}^m; z\right)$$

All necessary cancellations of terms $z^{C_{j,i}}$ in $Q\left(\mathbf{d}^m; z\right)$ are performed

$H\left(\mathbf{d}^m; z\right)$ has a pole $z = 1$ of order 1

$Q\left(\mathbf{d}^m; z\right)$ has a zero $z = 1$ of order $m - 1$

$\deg Q\left(\mathbf{d}^m; z\right) = F\left(\mathbf{d}^m\right) + \sigma_1 = \mathcal{Q}_m$

Gorenstein rings and complete intersection

Kunz (1970): *A semigroup ring* $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^m\right)\right]$ *is a Gorenstein ring if and only if a semigroup* $\mathsf{S}\left(\mathbf{d}^m\right)$ *is symmetric.*

Duality realation

$$Q\left(\mathbf{d}^m;z^{-1}\right)z^{\mathcal{Q}_m}=(-1)^{m-1}Q\left(\mathbf{d}^m;z\right),$$

$$\beta_k\left(\mathbf{d}^m\right)=\beta_{m-k-1}\left(\mathbf{d}^m\right),\qquad C_{j,k}+C_{j,m-k-1}=\mathcal{Q}_m$$

$\mathsf{S}\left(\mathbf{d}^m\right)$ is called *complete intersection (CI)* if $\beta_1=m-1$

$$Q\left(\mathbf{d}^m;z\right)=\left(1-z^{e_1}\right)\left(1-z^{e_2}\right)\ldots\left(1-z^{e_{m-1}}\right),$$

Every CI semigroup is also symmetric

Bresinsky (1975): *If* $m\geq 4$ *then* $\beta_1\left(\mathbf{d}^m\right)$ *is unbounded*

Symmetric semigroups $\mathsf{S}\left(\mathbf{d}^4\right)$ and $\mathsf{S}\left(\mathbf{d}^5\right)$

Bresinsky (1975): $\beta_1\left(\mathbf{d}^4\right)=3$ *if symmetric semigroup* $\mathsf{S}\left(\mathbf{d}^4\right)$ *is complete intersection, and* $\beta_1\left(\mathbf{d}^4\right)=5$ *if it is not*

Bresinsky (1979): *Let the generators* $\{d_j\}$ *of a symmetric semigroup* $\mathsf{S}(\left(\mathbf{d}^5\right)$ *satisfy the relation:*

$$d_i+d_j=d_k+d_l,\qquad \{i,j,k,l\}\in\{1,2,3,4,5\}$$

Then $\beta_1\left(\mathbf{d}^5\right)\leq 13$

Identities for degrees of syzygies in $\mathsf{S}(\mathbf{d}^m)$

LF (2009), Thm 1, Herzog-Kühl (1984), 1st part of Thm 1

Let $\mathsf{S}(\mathbf{d}^m)$ be given with its Hilbert series $H(\mathbf{d}^m; z)$. Then

$$1)\quad \sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^r - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^r + \cdots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^r = 0, \quad \text{if} \quad 1 \le r \le m-2,$$

$$2)\quad \sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^{m-1} - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^{m-1} + \cdots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^{m-1} = (-1)^m (m-1)! \pi_m$$

LF (2009), Thm 2

$$\Xi_q(\mathbf{d}^m) \subset \{d_1, \ldots, d_m\}, \quad \Xi_q(\mathbf{d}^m) = \{d_i \mid q \mid d_i\}, \quad \omega_q = \#\Xi_q(\mathbf{d}^m),$$

For every q, n such that $q \mid d_i$, $1 < q \le \max\{d_i\}$, and $\gcd(n, q) = 1$,

$$1)\quad \sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^r \exp\left(i\frac{2\pi n}{q} C_{j,1}\right) - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^r \exp\left(i\frac{2\pi n}{q} C_{j,2}\right) + \cdots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^r \exp\left(i\frac{2\pi n}{q} C_{j,m-1}\right) = 0, \quad \text{if} \quad 1 \le r \le \omega_q - 1$$

$$2)\quad \sum_{j=1}^{\beta_1(\mathbf{d}^m)} \exp\left(i\frac{2\pi n}{q} C_{j,1}\right) - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} \exp\left(i\frac{2\pi n}{q} C_{j,2}\right) + \cdots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} \exp\left(i\frac{2\pi n}{q} C_{j,m-1}\right) = 1$$

Applying identities

Complete intersections (CI) LF (2009)

$$\prod_{i=1}^{m-1} e_i = \pi_m, \qquad \mathcal{Q}_m \geq (m-1)\,\sqrt[m-1]{\pi_m}$$

Symmetric semigroups LF (2009)

$$\mathcal{B}_m = -1 + \beta_1(\mathbf{d}^m) - \beta_2(\mathbf{d}^m) + \cdots + (-1)^\gamma \beta_{\gamma-1}(\mathbf{d}^m)\,,\ \gamma = \lfloor m/2 \rfloor$$

If $edim = 2m$, then for $1 \leq k \leq 2m-1$

$$\mathcal{Q}_{2m}^k - \sum_{r=1}^{m-1} (-1)^r \sum_{j=1}^{\beta_r\left(\mathbf{d}^{2m}\right)} \left(C_{j,r}^k - [\mathcal{Q}_{2m} - C_{j,r}]^k \right) = (2m-1)!\pi_{2m}\delta_{k,2m-1}$$

If $edim = 2m+1$, then for $1 \leq k \leq 2m$

$$\mathcal{Q}_{2m+1}^k + \sum_{r=1}^{m-1} (-1)^r \sum_{j=1}^{\beta_r\left(\mathbf{d}^{2m+1}\right)} \left(C_{j,r}^k + [\mathcal{Q}_{2m+1} - C_{j,r}]^k \right) + (-1)^m \sum_{j=1}^{\beta_m\left(\mathbf{d}^{2m+1}\right)} C_{j,m}^k + (2m)!\, \pi_{2m+1}\, \delta_{k,2m} = 0$$

Symmetric (not CI) semigroups

Let $X_{1,m}=\sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}$

LF (2015) $\quad edim=4, \quad \beta_1=5$

$$X_{1,4}=2\mathcal{Q}_4, \qquad \mathcal{Q}_4\geq\sqrt[3]{25\pi_4}$$

LF (2017) $\quad edim=5, \quad \beta_2=2(\beta_1-1)$

$$1-\sqrt{1-\left(\frac{\mathcal{R}_5}{\mathcal{Q}_5}\right)^2}\leq\frac{2}{\beta_1}\frac{X_{1,5}}{\mathcal{Q}_5}\leq 1+\sqrt{1-\left(\frac{\mathcal{R}_5}{\mathcal{Q}_5}\right)^2}$$

$$\mathcal{Q}_5\geq\mathcal{R}_5=4\lambda_5\sqrt[4]{\pi_5}, \qquad \lambda_5=\sqrt[4]{\frac{3}{4}\frac{\beta_1-1}{\beta_1}}, \quad \delta=\frac{\mathcal{Q}_5}{\mathcal{R}_5}-1$$

S_5	A_1^6	A_2^6	A_1^7	A_2^7	A_1^8	A_2^8	A_1^9	A_2^9	A_1^{13}	A_2^{13}
β_1	6	6	7	7	8	8	9	9	13	13
$\mathcal{Q}_5$	90	92	87	93	99	89	102	96	240	236
$\mathcal{R}_5$	86.4	90.85	79.2	83.8	88.6	82.6	93.7	89.4	225.3	224.2
δ, %	4.16	1.26	9.84	10.9	11.7	7.74	8.85	7.38	6.52	5.26

$$\begin{array}{lll}
\beta_1=6, & A_1^6=\langle 10,11,12,13,15\rangle, & A_2^6=\langle 10,11,12,14,16\rangle,\\
\beta_1=7, & A_1^7=\langle 6,10,14,15,19\rangle, & A_2^7=\langle 6,10,14,17,21\rangle,\\
\beta_1=8, & A_1^8=\langle 6,10,14,19,23\rangle, & A_2^8=\langle 8,10,13,14,19\rangle,\\
\beta_1=9, & A_1^9=\langle 7,12,13,18,23\rangle, & A_2^9=\langle 9,12,13,14,19\rangle,\\
\beta_1=13, & A_1^{13}=\langle 19,23,29,31,37\rangle, & A_2^{13}=\langle 19,27,28,31,32\rangle.
\end{array}$$

Weak Asymptotics of Frobenius numbers $\mathbf{F}(\mathbf{d}^m)$

Define the asymptotics,

$$\langle \mathsf{A}(\mathbf{d}^m) \rangle = \lim_{N\to\infty} \sum_{\mathbf{d}^m \in \Delta_N^m} \frac{\mathsf{A}(\mathbf{d}^m)}{\#\Delta_N^m}, \quad \Delta_N^m = \{\mathbf{d}^m \mid m \le d_1, \dots, d_m \le N\}$$

Arnold's conjectures (1999):
For short, PDF - probability distribution function

1. $\langle c(\mathbf{d}^m)/\sqrt{\pi_m}\rangle = C_m, \qquad C_m = \sqrt[m-1]{(m-1)!}$??
2. $\langle \rho(\mathbf{d}^m) \rangle = 1/m$
3. Asymptotical PDF of filling the segment $[0, c(\mathbf{d}^m)]$ for large $\mathbf{d}^m$

$$\xi_m(s) = s^{m-1}/c(\mathbf{d}^m), \qquad \int \xi_3(s)ds = 1/m$$

LF (2006): *Conjectures 2, 3 fail for* $m = 3$*:*

$$4/9 < \langle \rho(\mathbf{d}^3) \rangle < 1/2$$

Bourgain, Sinai (2007):
There exists the uniform limiting distribution for the normalized Frobenius number

$$X_3 = c(\mathbf{d}^3)/\sqrt{\pi_3}$$

Ustinov (2008):

$$P(X_3) = \begin{cases} P_1(X_3) & if \quad 0 < X_3 < \sqrt{3}, \\ P_2(X_3) & if \quad \sqrt{3} < X_3 < 2, \\ P_3(X_3) & if \quad 2 < X_3 < \infty, \end{cases}$$

$$P_1(X_3) = 0, \qquad P_2(X_3) = \frac{12}{\pi}\, p_2(X_3), \qquad P_3(X_3) = \frac{12}{\pi^2}\, p_3(X_3),$$

$$p_2(X_3) = \frac{X_3}{\sqrt{3}} - \sqrt{4 - X^2}$$

$$p_3(X_3) = X\sqrt{3}\arccos\frac{X_3 + 3\sqrt{X_3^2 - 4}}{4\sqrt{X_3^2 - 3}} + \frac{3}{2}\sqrt{X_3^2 - 4}\log\frac{X_3^2 - 4}{X_3^2 - 3}$$

$$P(X_3) = \frac{18}{\pi^2}\frac{1}{X_3^3} + \mathcal{O}\left(\frac{1}{X_3^5}\right), \qquad X_3 \to \infty,$$

$$C_3 = \int_0^\infty X_3 P(X_3) dX_3 = \frac{8}{\pi}, \qquad \int_0^\infty P(X_3) dX_3 = 1$$

Marklof (2009), Strömbergsson (2011):

There exists the uniform limiting distribution for the normalized Frobenius number

$$X_m = c\left(\mathbf{d}^m\right) / \sqrt[m-1]{\pi_m}$$

$P(X_m)$ has almost all of its mass concentrated in the interval

$$\sqrt[m-1]{(m-1)!} < X_m < (1+\eta)\sqrt[m-1]{(m-1)!},$$

where $\eta + e\log\eta = 0$ and $\eta \simeq 0.756$

Asymptotics of $P(X_m)$

$$P(X_m) = \binom{m}{2}\frac{X_m^{-m}}{\zeta(m-1)} + \mathcal{O}\left(X_m^{-[m+1+1/(m-2)]}\right), \quad X_m \to \infty$$

The functions $P(X_m)$, $0 \leq X_m \leq \infty$, for $m \geq 4$ are still unknown !

Conjectures and questions

H. Wilf (1978)

$$\rho\left(\mathbf{d}^m\right) \geq \frac{1}{m}$$

$c\left(\mathbf{d}^m\right) \leq 2\mu\left(\mathbf{d}^m\right)$, *Kaplan* (2012), $c\left(\mathbf{d}^m\right) \leq 3\mu\left(\mathbf{d}^m\right)$, *Eliahou* (2016)

M. Bras-Amores (2008)

The number $N(g)$ of numerical semigroups of genus $g \leq 15$

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$N(g)$	1	1	2	4	7	12	23	39	67	118	204	343	592	1001	1693	2857

1) $N(g) \geq N(g-1) + N(g-2), \qquad g \geq 2$

2) $\lim\limits_{g\to\infty} \dfrac{N(g-1)+N(g-2)}{N(g)} = 1, \qquad$ *Zhai* (2013)

3) $\lim\limits_{g\to\infty} \dfrac{N(g)}{N(g-1)} = \phi, \qquad \phi = \dfrac{1+\sqrt{5}}{2}, \qquad$ *Zhai* (2013)

LF (2016)

$$F\left(\langle (n-1)^{2q}, n^{2q}, (n+1)^{2q}\rangle\right) = \mathcal{O}\left(n^{3q}\right)$$

$$F\left(\langle (n-1)^{2q+1}, n^{2q+1}, (n+1)^{2q+1}\rangle\right) = \mathcal{O}\left(n^{3q+2}\right)$$

$T_2 = 4, \quad T_3 = 18, \quad T_4 = 40, \quad T_{n\geq 5} = \ldots ?$
what are the congruence moduli T_k for the higher k ?

The problem comes up in many real-world contexts

Local Postage Stamp Problem

Having an unlimited supply of 17-cent, 18-cent, and 19-cent stamps we like to know what is the largest possible value of postage stamps on an envelope which we are unable to accomplish

Coin Change Problem

To determine the largest monetary amount that can not be obtained using only coins of specified denominations

Chicken McNugget Problem

Chicken McNuggets come in boxes of 6,9 and 20 nuggets. To find the largest number of nuggets that could not have been bought.

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THANK YOU FOR ATTENTION