# Frobenius Problem in Numerical Semigroups

*Expository talk*

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A numerical semigroup $\mathsf{S}(\mathbf{d}^m) = \langle d_1, \ldots, d_m \rangle$ is said to be generated by a minimal set of natural numbers $\mathbf{d}^m = \{d_1, \ldots, d_m\}$, $\gcd \mathbf{d}^m = 1$,

$$\mathsf{S}(\mathbf{d}^m) = \left\{ s \in \mathbb{N} \cup \{0\} \mid s = \sum_{i=1}^{m} x_i d_i,\ x_i \in \mathbb{N} \cup \{0\} \right\},$$

if neither of its elements is linearly representable by the rest of them. F. G. Frobenius in his lectures repeatedly raised the question of determining (or bounding) the largest integer $F(\mathbf{d}^m)$ which is unrepresentable by the tuple $\{d_1, \ldots, d_m\}$. This number appears in various contexts in combinatorics, partition theory, commutative algebra, algebraic geometry, probability theory and computer science. In the last decades a research activity in this area was especially fruitful. In this talk we discuss basic results and bring to attention some conjectures.

The talk is expository. We give basic notions and terminology relating to $\mathsf{S}(\mathbf{d}^m)$. A standard classification of semigroups presumes their separation into two sets: non-symmetric and symmetric, including complete intersection (CI). Such dichotomy becomes clear when we associate with $\mathsf{S}(\mathbf{d}^m)$ a semigroup ring $\mathsf{k}[\mathsf{S}(\mathbf{d}^m)]$ with its minimal free resolution and Hilbert's series $H(\mathbf{d}^m; z) = \sum_{s \in \mathsf{S}(\mathbf{d}^m)} z^s$. We present various special semigroups allowing to calculate $F(\mathbf{d}^m)$, in particular, $\mathsf{S}(\mathbf{d}^2)$ and $\mathsf{S}(\mathbf{d}^3)$. In general case of $\mathsf{S}(\mathbf{d}^m)$, we give two recent theorems for degrees $C_{ij}$ of syzygies and show many applications that give rise to new relations between the generators $d_j$, Betti's numbers $\beta_k$ and degrees $C_{ij}$. The most striking of them are the bounds for $F(\mathbf{d}^4)$ and $F(\mathbf{d}^5)$ in symmetric (not CI) semigroups. We discuss also the recent progress in solving the Arnold's conjectures for distribution of normalized Frobenius numbers $F(\mathbf{d}^m) / \sqrt[m-1]{\pi_m}$, where $\pi_m = \prod_{i=1}^m d_i$ and $m \le d_i \le N$, $N \to \infty$.