Frobenius Problem
in Numerical Semigroups

Expository talk

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CONTENT

1. Definition and notations

2. Frobenius problem – basic facts

3. Special numerical semigroups $S(d^m)$

4. Symmetric numerical semigroups $S(d^m)$

5. Semigroups $S(d^3)$ and minimal relations

6. Semigroup rings $k[S(d^m)]$

7. Gorenstein rings and complete intersection

8. Identities for degrees of syzygies in $S(d^m)$

9. Applying identities

10. Weak Asymptotics of Frobenius numbers $F(d^3)$

11. Conjectures and questions
Numerical semigroups: definition

A numerical semigroup $S(d^m) = \langle d_1, \ldots, d_m \rangle$ is said to be generated by a minimal set of natural numbers $d^m = \{d_1, \ldots, d_m\}$, $\gcd d^m = 1$

$$S(d^m) = \left\{ s \in \mathbb{N} \cup \{0\} \mid s = \sum_{i=1}^{m} x_i d_i, \ x_i \in \mathbb{N} \cup \{0\} \right\},$$

if neither of its elements is linearly representable by the rest of them.

Notations

$d_1, \ldots, d_m$ - generators, $\pi_m = \prod_{i=1}^{m} d_i$, $\sigma_1 = \sum_{i=1}^{m} d_i$

$\mu(d^m) = \min\{d_1, \ldots, d_m\}$ - a multiplicity,

$m$ - an embedding dimension, $edim$,

$\mathbb{N} \setminus S(d^m)$ - a set of gaps of semigroup,

$F(d^m) = \max\{\mathbb{N} \setminus S(d^m)\}$ - the Frobenius number,

$g(d^m) = \#\{\mathbb{N} \setminus S(d^m)\}$ - a genus of semigroup,

$c(d^m) = 1 + F(d^m) - a conductor of semigroup,$

$$c(d^m) \leq 2g(d^m), \quad Q_m = F(d^m) + \sigma_1$$

$\rho(d^m) = 1 - g(d^m) / c(d^m)$ - a density of non-gaps

$Ap[S(d^m); a]$ - the Apéry set of $S(d^m)$ w.r.t.

$H(d^m; z) = \sum_{s \in S(d^m)} z^s, \ z < 1$, - the Hilbert series

$\Phi(d^m; z) = \sum_{s \notin S(d^m)} z^s$ - a generating function of gaps

$$H(d^m; z) + \Phi(d^m; z) = \frac{1}{1 - z}, \quad z < 1$$
Frobenius problem – basic facts

According to oral (unpublished) reminiscences of I. Schur,
"... G. Frobenius in his lectures repeatedly raised the question of determining (or bounding) $F(d^m)$"

A. Brauer (1942): The Theorems in §3-5 result partly from discussions of Schur and the author. It was formerly intended to publish these results in a joint paper. I conform with Schur’s wishes that the publishing be not longer postponed and that I publish the paper alone.

The paper was submitted for publication in 1940, less than two months before Schurs death, and was published two years later.

Curtis (1990): There is no polynomial $\Psi \in \mathbb{C}[X_1, \ldots, X_m, Y]$, $\Psi \not\equiv 0$, such that

$$\Psi(d_1, \ldots, d_m, F(d^m)) = 0,$$

for each choice of $d^m$, $m \geq 3$

Kannan (1992): There exists a polynomial time algorithm for any fixed $m$ but the time depends at least exponentially on $m$

Ramírez-Alfonsín (1996): The Frobenius problem is NP-hard under Turing-reductions

Applications

Combinatorics, Number theory, Commutative algebra, Algebraic Geometry, Probability, Computer science
Special numerical semigroups \( S(d^m) \)

Roberts (1956): Arithmetic sequence

\[ d^m = \{a, a + d, \ldots, a + (m - 1)d\} \]

\[ F(d^m) = a \left( \left\lfloor \frac{a - 2}{m - 1} \right\rfloor + 1 \right) + (d - 1)(a - 1) - 1 \]

Selmer (1997), Rødseth (1999): Almost arithmetic sequence

\[ d^m = \{a, ha + d, ha + 2d, \ldots, ha + (m - 1)d\} \]

\[ F(d^m) = ha \left\lfloor \frac{a - 2}{m - 1} \right\rfloor + a(h - 1) + d(a - 1) \]

\[ d^m = \{a, a + 1, a + 2, a + 2^2, \ldots, a + 2^{m-2}\} \]

\[ F(d^m) = \frac{a(a + 1)}{2^{m-2}} + \sum_{k=0}^{m-3} 2^k \left\lfloor \frac{a + 2^k}{2^{m-2}} \right\rfloor + a(m - 4) - 1 \]

Ong, Ponomarenko (2008): Geometric sequence

\[ d^m = \{a^{m-1}, a^{m-2}b, a^{m-3}b^2, \ldots, b^{m-1}\}, \quad b > a, \quad \gcd(a, b) = 1 \]

\[ F(d^m) = b^{m-2}(ab - a - b) + \frac{(b - 1)a^2(a^{m-2} - b^{m-2})}{a - b} \]

The Apéry set of \( S(d^m) \) w.r.t. \( a \in S(d^m), a \neq 0 \)

\[ Ap[S(d^m); a] = \{s \in S(d^m) \mid s - a \not\in S(d^m)\} \]

Schockley, Brauer (1962)

\[ \#Ap[S(d^m); a] = a, \quad F(d^m) = \max Ap[S(d^m); a] - a \]

Example: \( Ap[\langle 7, 12, 13 \rangle, 7] = \{0, 12, 13, 24, 25, 36, 37\}, \quad F = 30 \)
Symmetric semigroups

A semigroup $S(d^m)$ is called symmetric if for any integer $s$ the following condition holds: if $s \in S(d^m)$ then $F(d^m) - s \not\in S(d^m)$. Otherwise $S(d^m)$ is called nonsymmetric.

$$c(d^m) = 2g(d^m), \quad \rho(d^m) = \frac{1}{2}$$

If $\mu(d^m) = m$ then semigroup $S(d^m)$ is never symmetric.

Examples

$S_1 = \langle 4, 7 \rangle = \{0, 4, 7, 8, 11, 12, 14, 15, 16, 18, \mapsto \}$,
$G_1 = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}, \quad F_1 = 17, \quad c_1 = 18, \quad g_1 = 9,$
$S_2 = \langle 4, 5, 6, 7 \rangle = \{0, 4, \mapsto \}$,
$G_2 = \{1, 2, 3\}, \quad F_2 = 3, \quad c_2 = 4, \quad g_2 = 3, \quad \rho = 1/4$

Watanabe (1973): Let $H_1 = \langle d_1, \ldots, d_m \rangle$ be a semigroup, and $a, b \in \mathbb{Z}_+$ such that:

- $a \in H_1 \setminus \{d_1, \ldots, d_m\}$,
- $\gcd(a, b) = 1$

Denote by $H = \langle a, bH_1 \rangle$ a semigroup $H = \langle a, bd_1, \ldots, bd_k \rangle$, then $H$ is symmetric if and only if $H_1$ is symmetric.

Example

$H = \langle 20, 21, 24, 27, 39 \rangle, \quad H_1 = \langle 7, 8, 9, 13 \rangle$, $20, \quad b = 3$

Johnson (1960):

$$F(H) = bF(H_1) + (b - 1)a$$
Telescopic semigroups

$S(d^m)$ is said to be telescopic iff for all $2 \leq k \leq m$,
\[
\frac{d_k}{g_k} \in S_{k-1}, \quad S_k = S\left(\left\{\frac{d_1}{g_k}, \frac{d_2}{g_k}, \ldots, \frac{d_k}{g_k}\right\}\right), \quad g_k = \gcd(d_1, \ldots, d_k) > 1
\]

Example $H = \langle 11, 5(2, 3), 7, 17 \rangle = \langle 17, 77, 110, 165 \rangle$

Delorme (1976): Let $H_1 = \langle d_1, \ldots, d_m \rangle$ and $H_2 = \langle c_1, \ldots, c_k \rangle$ be semigroups, and $a, b \in \mathbb{Z}_+$ such that:

- $a \in H_1 \setminus \{d_1, \ldots, d_m\}$,  
- $b \in H_2 \setminus \{c_1, \ldots, c_k\}$,  
- $\gcd(a, b) = 1$

Denote by $H = \langle bH_1, aH_2 \rangle = \langle bd_1, \ldots, bd_m, ac_1, \ldots, ac_k \rangle$, then $H$ is symmetric if and only if $H_1$ and $H_2$ are symmetric.

Example

$H = \langle 52, 54, 65, 63 \rangle$  
$H_1 = \langle 4, 5 \rangle$  
$H_2 = \langle 6, 7 \rangle$  
$a = 9$  
$b = 13$

Semigroups $S(d^2)$

The semigroup $S(d^2)$ is always symmetric

Sylvester (1884), Rödseth (1994):

$F(d^2) = d_1d_2 - d_1 - d_2$,  
$g(d^2) = \frac{(d_1 - 1)(d_2 - 1)}{2}$,

$H(d^2, z) = \frac{1 - z^{d_1d_2}}{(1 - z^{d_1})(1 - z^{d_2})}$,  
$g_n(d^2) = \sum_{s \text{ gaps}} s^n$,

$g_n(d^2) = K_n \sum_{k=0}^{n+1} \sum_{l=0}^{n+1-k} \binom{n+2}{k} \binom{n+2-k}{l} B_k B_l d_1^{n+1-k} d_2^{n+1-l} - B_{n+1}/(n+1)$,  
$K_n = ((n+1)(n+2))^{-1}$
Pseudo-symmetric semigroups

A numerical semigroup $S(d^m)$ is said to be pseudo-symmetric if

$$c(d^m) = 2g(d^m) - 1$$

Arf semigroups

A numerical semigroup $S(d^m)$ is called Arf semigroup if

$$s_i + s_j - s_k \in S(d^m) \text{ for every } i, j, k \in \mathbb{N} \text{ and } i \geq j \geq k$$

Munuera, Torres, Villanueva (2009)

Arf semigroups are sparse. Let $S(d^m)$ is Arf semigroup and

$s_i, s_{i+1} < c(d^m)$ are two non-gaps. Then $s_{i+1} - s_i \geq 2$

The numerical semigroup is Arf iff for every two positive integers $i, j$ with $i > j$ it holds $2s_i - s_j \in S(d^m)$

Hyperelliptic semigroups

A semigroup $S(d^2)$ is called hyperelliptic if it is generated by 2 and an odd integer $2k + 1$. It is of the form

$$S(d^m) = \{0, 2, 4, \ldots, 2k - 2, 2k, \rightarrow\}$$

Campillo, Farran, Munuera, (2000)

The only Arf symmetric semigroups are hyperelliptic semigroups

Bras-Amores (2016)

The only Arf pseudo-symmetric semigroups are $\{0, 3, \rightarrow\}$ and $\{0, 3, 5, \rightarrow\}$
Semigroups $S_3(d^3)$ and matrix of minimal relations

Let a nonsymmetric semigroup $S_3 = \langle d_1, d_2, d_3 \rangle$ be given by matrix of minimal relations, $A_3$, $a_{ij} \in \mathbb{Z}_+$,

$$A_3 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix},$$

$$a_{11} = \min \{v_{11} \mid v_{11} \geq 2, \ v_{11}d_1 = v_{12}d_2 + v_{13}d_3, \ v_{12}, v_{13} \in \mathbb{N} \cup \{0\}\}$$

$$a_{22} = \min \{v_{22} \mid v_{22} \geq 2, \ v_{22}d_2 = v_{21}d_1 + v_{23}d_3, \ v_{21}, v_{23} \in \mathbb{N} \cup \{0\}\}$$

$$a_{33} = \min \{v_{33} \mid v_{33} \geq 2, \ v_{33}d_3 = v_{31}d_1 + v_{32}d_2, \ v_{31}, v_{32} \in \mathbb{N} \cup \{0\}\}$$

All matrix elements $a_{ij}$ are non-negative integers such that

$$a_{11} = a_{21} + a_{31}, \quad a_{22} = a_{12} + a_{32}, \quad a_{33} = a_{13} + a_{23},$$

$$d_1 = a_{22}a_{33} - a_{23}a_{32}, \quad d_2 = a_{33}a_{11} - a_{31}a_{13},$$

$$d_3 = a_{11}a_{22} - a_{12}a_{21}.$$ 

Hilbert series $H(z, S_3)$, Frobenius number $F(S_3)$, genus $g(S_3)$

$$H(z, S_3) = \frac{1 - z^{e_1} - z^{e_2} - z^{e_3} + z^{t_1} + z^{t_2}}{(1 - z^{d_1})(1 - z^{d_2})(1 - z^{d_3})},$$

$$e_i = a_{ii}d_i, \quad t_1 = \delta_0 + \delta_1, \quad t_2 = \delta_0 + \delta_2,$$

$$\delta_0 = a_{11}a_{22}a_{33}, \quad \delta_1 = a_{12}a_{23}a_{31}, \quad \delta_2 = a_{13}a_{32}a_{21},$$

$$e_1 + e_2 + e_3 = t_1 + t_2, \quad F_1 = t_1 - \sigma_1, \quad F_2 = t_2 - \sigma_1,$$

$$F(S_3) = \max \{F_1, F_2\}, \quad 2g(S_3) = 1 + \delta_0 + \delta_1 + \delta_2 - \sigma_1$$
Let a nonsymmetric numerical semigroup $S_3$ be given. Then the matrix $A_3$ of minimal relations has a unique Rep.

Rosales, Gárcia-Sánchezel LF (2004):

\[ a = \{a_{11}, a_{22}, a_{33}\}, \quad \langle a, d^3 \rangle = a_{11}d_1 + a_{22}d_2 + a_{33}d_3, \]

\[ g(d^m) = \frac{1}{2} \left[ 1 + \langle a, d^3 \rangle - \sigma_1 - \delta_0 \right], \]

\[ F(d^3) = \frac{1}{2} \left[ \langle a, d^3 \rangle + J(d^3) \right] - \sigma_1, \]

\[ J^2(d^3) = \langle a, d^3 \rangle^2 - 4 \sum_{i>j} a_{ii}a_{jj}d_id_j + 4\pi_3 \]

**Lower and Upper Bounds**

Rödseth (1990), Davison (1994):

\[ Q_3 > \sqrt{3}\sqrt{\pi_3}, \]

LF (2004):

\[ Q_3 > \sqrt{3}\sqrt{\pi_3} + 1 \]

Beck, Einstein, Zack (BEZ) conjecture (2003):

Based on the thousands randomly chosen admissible triples $(d_1, d_2, d_3)$ satisfying the condition $\sqrt{\pi_3} < 2 \times 10^4$

\[ Q_3 < C\pi_3^\nu, \quad \nu < 2/3 \]

LF (2004): $Q_3$ cannot be bounded by any $R_{BEZ}$ given by

\[ Q_{BEZ} = C\pi_3^\nu \]

for any $\nu < 2/3$ and $C > 0$
Short proof:

Consider a semigroup $S_3 = \langle 2l + 1, 2l + 3, 4l + 3 \rangle$ with matrix $A_3$ of minimal relations,

$$A_3 = \begin{pmatrix} l + 3 & -l & -1 \\ -l & l + 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}, \quad \gcd(2l + 1, 2l + 3, 4l + 3) = 1,$$

$$F(S_3) = 2l^2 + 3l - 1, \quad Q_3 = 2l^2 + 11l + 6$$

Denote by $\delta_{C;\nu}(l)$ the ratio

$$\delta_{C;\nu}(l) = \frac{Q_{BEZ}}{Q_3}$$

and note that the leading asymptotic term of $\delta_{C;\nu}(l)$ when $l \gg 1$ reads,

$$\delta_{C;\nu}(l) \approx C 2^{4\nu - 1} l^{3\nu - 2}$$

and leads to refuting the BEZ conjecture:

$$\delta_{C;\nu}(l) < 1 \quad \text{if} \quad l > l_{cr} \quad \text{where} \quad \log_2 l_{cr} = \frac{4\nu - 1}{2 - 3\nu} + \frac{\log_2 C}{2 - 3\nu}$$

**Example:** $\nu = 5/8, \quad C = 1, \quad l_{cr} = 4096, \quad l = 5000$

$d_1 = 10001 = 73 \cdot 137, \quad d_2 = 10003 = 7 \cdot 1429, \quad d_3 = 20003 = 83 \cdot 241,$

$$Q_3 = 50014999, \quad Q_{BEZ} = 48745746.76$$

**Example:** $\nu = 5/8, \quad C = 1.35, \quad l_{cr} = 45189, \quad l = 50000$

$d_1 = 100001 = 11 \cdot 9091, \quad d_2 = 100003, \quad d_3 = 200003,$

$$Q_3 = 5000149999, \quad Q_{BEZ} = 3656883908.53$$
Semigroup series


\[ g_n (d^3) = \sum_{s \text{ gaps}} s^n, \quad n \geq 0, \]

\[ g_0 (d^3) = \frac{1}{2} \left[ 1 - \sigma_1 - \delta_0 + \langle a, d^3 \rangle \right], \]

\[ g_1 (d^3) = \frac{1}{12} \left[ -1 + \pi_3 + \sum_{i=1}^{3} A_i d_i^2 + \sum_{i>j}^{3} B_{ij} d_i d_j - \delta_0 \sum_{i=1}^{3} C_i d_i \right], \]

\[ A_i = (a_{ii} - 1)(2a_{ii} - 1), \quad C_i = 2a_{ii} - 3, \]

\[ B_{ij} = 3(a_{ii} - 1)(a_{jj} - 1) - a_{ii} a_{jj} \]

The other formulas for \( n \geq 2 \) are too long to be presented here.

LF, Komatsu (2017)

\[ g_n (d^2) = \sum_{s \in \mathbb{N} \setminus S_2} s^{-n}, \quad n \geq 1, \quad S_2 = \langle d_1, d_2 \rangle, \]

\[ g_{-n}(S_2) = \left( 1 - \frac{1}{d_2^n} \right) \zeta(n) - \frac{1}{d_2^n} \sum_{k=1}^{d_2-1} \zeta \left( n, \frac{k d_1}{d_2} \right), \]

where \( \zeta(n, q) \) and \( \zeta(n) \) stand for the Hurwitz and Riemann zeta functions,

\[ \zeta(n, q) = \sum_{k=0}^{\infty} \frac{1}{(k + q)^n}, \quad \zeta(n, 1) = \zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}. \]
Special Numerical Semigroups $S(d^3)$

Kraft (1985): Pythagorean semigroup: $d_1^2 + d_2^2 = d_3^2$

$$\langle d_1^2 - d_2^2, 2d_1d_2, d_1^2 + d_2^2 \rangle$$

$$F(d^3) = k_1(k_1^2 - k_2^2 + 2(k_1k_2 - k_1 - k_2))$$

Marin, Ramirez Alfonsin, Revuelta (2007): Fibonacci semigroup

$$\langle \{F_i, F_{i+1}, F_{i+k}\}, \ i, k \geq 3, \quad r = \lfloor (F_i - 1)/F_k \rfloor \rangle$$

$$F(d^3) = \begin{cases} (F_i - 1)F_{i+2} - F_i(rF_{k-2} + 1) & \text{if } C_1, \\ (rF_k - 1)F_{i+2} - F_i((r - 1)F_{k-2} + 1) & \text{if } C_2, \end{cases}$$

$C_1$: if $r = 0$ or $r \geq 1$ and $F_{k-2}F_i < (F_i - rF_k)F_{i+2}$

$C_2$: otherwise

LF (2007): Lucas symmetric semigroup

$$\langle L_k, L_m, L_n \rangle, \quad k, m, n \geq 2$$

Lepilov, O’Rourke, Swanson (2015)

$$\langle (n-1)^k, n^k, (n+1)^2 \rangle, \quad n \neq 3, 4, 5 \quad \text{congruence modulo } T_2 = 4$$

$$\langle (n-1)^3, n^3, (n+1)^3 \rangle, \quad n \neq 3 \quad \text{congruence modulo } T_3 = 18$$

LF (2016)

$$\langle (n-1)^4, n^4, (n+1)^4 \rangle, \quad n \neq 3, 5, 7 \quad \text{congruence modulo } T_4 = 40$$
LF (2016)

Matrix $A_3$ for semigroups $\langle (n - 1)^2, n^2, (n + 1)^2 \rangle$:

$$
\begin{pmatrix}
16m + 1 & -8(2m - 1) & -1 \\
-(15m + 1) & 16m - 7 & -(m - 1) \\
-m & -1 & m
\end{pmatrix}, \quad n = 4m
$$

$$
G = 4m(34m^2 - 21m + 2),
F = 20, \quad m = 1,
F = 272m^3 - 168m^2 + m - 2, \quad m \geq 2.
$$

$$
\begin{pmatrix}
7m + 2 & -4(m - 1) & -(3m + 1) \\
-(3m + 1) & 4m & -m \\
-(4m + 1) & -4 & 4m + 1
\end{pmatrix}, \quad n = 4m + 1
$$

$$
G = 2m(32m^2 + 9m + 1),
F = 112m^3 + 48m^2 + 8m - 1, \quad m \leq 3,
F = 128m^3 - 20m - 5, \quad m \geq 4.
$$

$$
\begin{pmatrix}
9m + 5 & -(8m - 1) & -(m + 1) \\
-(7m + 4) & 8m + 1 & -m \\
-(2m + 1) & -2 & 2m + 1
\end{pmatrix}, \quad n = 4m + 2
$$

$$
G = m(80m^2 + 71m + 16),
F = 312, \quad m = 1,
F = 160m^3 + 128m^2 + 10m - 9, \quad m \geq 2.
$$

$$
\begin{pmatrix}
5m + 4 & -4m & -(m + 1) \\
-(m + 1) & 4(m + 1) & -(3m + 2) \\
-(4m + 3) & -4 & 4m + 3
\end{pmatrix}, \quad n = 4m + 3
$$

$$
G = 2(32Nm^3 + 57m^2 + 33m + 6),
F = 128m^3 + 2242 + 124m + 19, \quad m \geq 1.
$$
Semigroup Rings

Let \( R = k[X_1, \ldots, X_m] \) be a polynomial ring over a field \( k \) of characteristic 0 and \( \psi_0 : R \rightarrow k[z^{d_1}, \ldots, z^{d_m}] \) be the projection induced by \( X_i = z^{d_i} \).

A semigroup ring \( k[S(d^m)] = k[z^{d_1}, \ldots, z^{d_m}] \) is a graded 1-dim local subring of \( R \) and has a presentation

\[
k[S(d^m)] = R/M_1, \quad M_1 = \ker(\psi_0)
\]

and is associated to affine monomial curve with parametrization

\[
X_1 = z^{d_1}, \quad X_2 = z^{d_2}, \quad \ldots, \quad X_m = z^{d_m}
\]

A graded \( R \)-module \( M_1 \) is minimally generated by a finite number \( \beta_1 \) of binomial generators \( P_k, 1 \leq k \leq \beta_1, \)

\[
P_k(X_1, \ldots, X_m) = X_1^{h_{i_1}^k} \cdots X_m^{h_{i_m}^k} - X_1^{q_{j_1}^k} \cdots X_m^{q_{j_m}^k},
\]

\( h_{i_j}^k, q_j^k \in \mathbb{Z}_{\geq 0} \), such that \( \psi_0(P_k) = 0 \). Associated arithmetic relations

\[
\sum_{i=1}^{m} h_{i}^k d_i = \sum_{j=1}^{m} q_j^k d_j, \quad C_{1,k} = \sum_{i=1}^{m} h_{i}^k d_i,
\]

are called syzygies of the 1st kind with degrees \( C_{1,k} \) and \( M_1 \) is called the 1st syzygy module of \( k[S(d^m)] \), or ideal.

For \( n \geq 2 \) it may happen that any set of \( P_k \in M_1 \) has their own relations. It give rise to existence of another free module \( R_1 \), such that

\[
R_1 \xrightarrow{\psi_1} R \xrightarrow{\psi_0} k[S(d^m)], \quad \operatorname{Im}(\psi_1) = M_1 = \ker(\psi_0)
\]
Going to the higher syzygies we arrive at the Hilbert Syzygy Theorem

Hilbert’s Syzygy Theorem: Let \( k[S(d^m)] \) be a finitely generated module over the polynomial ring \( R = k[X_1, \ldots, X_m] \). Then there is a minimal free resolution

\[
0 \rightarrow R_n \xrightarrow{\psi_n} \ldots \xrightarrow{\psi_2} R_1 \xrightarrow{\psi_1} R \xrightarrow{\psi_0} k[S(d^m)] \rightarrow 0,
\]

\( \text{Im}(\psi_k) = M_k = \ker(\psi_{k-1}) \), \( 1 \leq k \leq n \)

where \( R_1, \ldots, R_n \) are finitely generated free \( R \)-modules and \( n < m \).

If \( k[S(d^m)] \) is a graded 1-dim local subring of \( R \) then \( n = m - 1 \)

An integer \( \beta_k = \dim M_k \) is called the \( k \)-th Betti number

\[
1 - \beta_1 + \beta_2 - \ldots + (-1)^{m-1}\beta_{m-1} = 0, \quad \beta_1 \geq m - 1
\]

The Hilbert series

\[
H(d^m; z) = \frac{Q(d^m; z)}{\prod_{i=1}^{m} (1 - z^{d_i})},
\]

\[
Q(d^m; z) = 1 - Q_1(d^m; z) + \cdots + (-1)^{m-1}Q_{m-1}(d^m; z),
\]

\[
Q_i(d^m; z) = \sum_{j=1}^{\beta_i(d^m)} z^{C_{j,i}}, \quad \deg Q_i(d^m; z) < \deg Q_{i+1}(d^m; z)
\]

All necessary cancellations of terms \( z^{C_{j,i}} \) in \( Q(d^m; z) \) are performed

\( H(d^m; z) \) has a pole \( z = 1 \) of order 1

\( Q(d^m; z) \) has a zero \( z = 1 \) of order \( m - 1 \)

\( \deg Q(d^m; z) = F(d^m) + \sigma_1 = Q_m \)
Gorenstein rings and complete intersection

Kunz (1970): A semigroup ring \( k[S(d^m)] \) is a Gorenstein ring if and only if a semigroup \( S(d^m) \) is symmetric.

Duality realation

\[
Q(d^m; z^{-1}) z^{Q_m} = (-1)^{m-1} Q(d^m; z),
\]

\[
\beta_k(d^m) = \beta_{m-k-1}(d^m), \quad C_{j,k} + C_{j,m-k-1} = Q_m
\]

\( S(d^m) \) is called complete intersection (CI) if \( \beta_1 = m - 1 \)

\[
Q(d^m; z) = (1 - z^{e_1})(1 - z^{e_2}) \ldots (1 - z^{e_{m-1}}),
\]

Every CI semigroup is also symmetric

Bresinsky (1975): If \( m \geq 4 \) then \( \beta_1(d^m) \) is unbounded

Symmetric semigroups \( S(d^4) \) and \( S(d^5) \)

Bresinsky (1975): \( \beta_1(d^4) = 3 \) if symmetric semigroup \( S(d^4) \) is complete intersection, and \( \beta_1(d^4) = 5 \) if it is not

Bresinsky (1979): Let the generators \( \{d_j\} \) of a symmetric semigroup \( S((d^5)) \) satisfy the relation:

\[
d_i + d_j = d_k + d_l, \quad \{i, j, k, l\} \in \{1, 2, 3, 4, 5\}
\]

Then \( \beta_1(d^5) \leq 13 \)
Identities for degrees of syzygies in $S(d^m)$

LF (2009), Thm 1, Herzog-Kühl (1984), 1st part of Thm 1

Let $S(d^m)$ be given with its Hilbert series $H(d^m; z)$. Then

1) $\sum_{j=1}^{\beta_1(d^m)} C_{j,1}^r - \sum_{j=1}^{\beta_2(d^m)} C_{j,2}^r + \cdots + (-1)^m \sum_{j=1}^{\beta_{m-1}(d^m)} C_{j,m-1}^r = 0,$
   \[ \text{if } 1 \leq r \leq m - 2, \]

2) $\sum_{j=1}^{\beta_1(d^m)} C_{j,1}^{m-1} - \sum_{j=1}^{\beta_2(d^m)} C_{j,2}^{m-1} + \cdots + (-1)^m \sum_{j=1}^{\beta_{m-1}(d^m)} C_{j,m-1}^{m-1} =$
   \[ (-1)^m (m-1)! \pi_m \]

LF (2009), Thm 2

$\Xi_q (d^m) \subset \{ d_1, \ldots, d_m \}$, $\Xi_q (d^m) = \{ d_i \mid q \mid d_i \}$, $\omega_q = \# \Xi_q (d^m)$.

For every $q, n$ such that $q \mid d_i$, $1 < q \leq \max \{ d_i \}$, and $\gcd(n, q) = 1$,

1) $\sum_{j=1}^{\beta_1(d^m)} C_{j,1}^r \exp \left( i \frac{2\pi n}{q} C_{j,1}^r \right) - \sum_{j=1}^{\beta_2(d^m)} C_{j,2}^r \exp \left( i \frac{2\pi n}{q} C_{j,2}^r \right) + \cdots + (-1)^m \sum_{j=1}^{\beta_{m-1}(d^m)} C_{j,m-1}^r \exp \left( i \frac{2\pi n}{q} C_{j,m-1}^r \right) = 0,$
   \[ \text{if } 1 \leq r \leq \omega_q - 1 \]

2) $\sum_{j=1}^{\beta_1(d^m)} \exp \left( i \frac{2\pi n}{q} C_{j,1}^r \right) - \sum_{j=1}^{\beta_2(d^m)} \exp \left( i \frac{2\pi n}{q} C_{j,2}^r \right) + \cdots + (-1)^m \sum_{j=1}^{\beta_{m-1}(d^m)} \exp \left( i \frac{2\pi n}{q} C_{j,m-1}^r \right) = 1$
Applying identities

Complete intersections (CI) LF (2009)

\[
\prod_{i=1}^{m-1} e_i = \pi_m, \quad Q_m \geq (m - 1)^{m-\frac{1}{\sqrt{\pi_m}}}
\]

Symmetric semigroups LF (2009)

\[
B_m = -1 + \beta_1 (d^m) - \beta_2 (d^m) + \cdots + (-1)^\gamma \beta_{\gamma-1} (d^m), \quad \gamma = \lfloor m/2 \rfloor
\]

If \( edim = 2m \), then for \( 1 \leq k \leq 2m - 1 \)

\[
Q_{2m}^k - \sum_{r=1}^{m-1} (-1)^r \sum_{j=1}^{\beta_r(d^{2m})} \left( C_{j,r}^k - [Q_{2m} - C_{j,r}]^k \right) = (2m - 1)! \pi_{2m} \delta_{k,2m-1}
\]

If \( edim = 2m + 1 \), then for \( 1 \leq k \leq 2m \)

\[
Q_{2m+1}^k + \sum_{r=1}^{m-1} (-1)^r \sum_{j=1}^{\beta_r(d^{2m+1})} \left( C_{j,r}^k + [Q_{2m+1} - C_{j,r}]^k \right) + \beta_m(d^{2m+1})
\]

\[
(-1)^m \sum_{j=1}^{\beta_m(d^{2m+1})} C_{j,m}^k + (2m)! \pi_{2m+1} \delta_{k,2m} = 0
\]
Symmetric (not CI) semigroups

Let $X_{1,m} = \sum_{j=1}^{\beta_1 m} C_{j,1}$

**LF (2015)** $edim = 4, \quad \beta_1 = 5$

$$X_{1,4} = 2Q_4, \quad Q_4 \geq \sqrt[3]{25\pi_4}$$

**LF (2017)** $edim = 5, \quad \beta_2 = 2(\beta_1 - 1)$

$$1 - \sqrt{1 - \left(\frac{R_5}{Q_5}\right)^2} \leq \frac{2}{\beta_1} \frac{X_{1,5}}{Q_5} \leq 1 + \sqrt{1 - \left(\frac{R_5}{Q_5}\right)^2}$$

$$Q_5 \geq R_5 = 4\lambda_5 \sqrt[4]{\pi_5}, \quad \lambda_5 = \sqrt[4]{\frac{3\beta_1 - 1}{4\beta_1}}, \quad \delta = \frac{Q_5}{R_5} - 1$$

<table>
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<th>$\beta_2$</th>
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<th>$A_{12}^6$</th>
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<td>224.2</td>
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<td>7.38</td>
<td>6.52</td>
<td>5.26</td>
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</table>

$\delta, \%$ 4.16 1.26 9.84 10.9 11.7 7.74 8.85 7.38 6.52 5.26

$\beta_1 = 6, \quad A_{11}^6 = \langle 10, 11, 12, 13, 15 \rangle, \quad A_{12}^6 = \langle 10, 11, 12, 14, 16 \rangle,$

$\beta_1 = 7, \quad A_{11}^7 = \langle 6, 10, 14, 15, 19 \rangle, \quad A_{12}^7 = \langle 6, 10, 14, 17, 21 \rangle,$

$\beta_1 = 8, \quad A_{11}^8 = \langle 6, 10, 14, 19, 23 \rangle, \quad A_{12}^8 = \langle 8, 10, 13, 14, 19 \rangle,$

$\beta_1 = 9, \quad A_{11}^9 = \langle 7, 12, 13, 18, 23 \rangle, \quad A_{12}^9 = \langle 9, 12, 13, 14, 19 \rangle,$

$\beta_1 = 13, \quad A_{11}^{13} = \langle 19, 23, 29, 31, 37 \rangle, \quad A_{12}^{13} = \langle 19, 27, 28, 31, 32 \rangle.$
Weak Asymptotics of Frobenius numbers $F(d^m)$

Define the asymptotics,

$$\langle A(d^m) \rangle = \lim_{N \to \infty} \sum_{d^m \in \Delta^m_N} \frac{A(d^m)}{\# \Delta^m_N}, \quad \Delta^m_N = \{d^m \mid m \leq d_1, \ldots, d_m \leq N\}$$

Arnold’s conjectures (1999):
For short, PDF - probability distribution function

1. $\langle c(d^m) / \sqrt{\pi m} \rangle = C_m, \quad C_m = \frac{m^{-1}}{\sqrt{(m-1)!}}$ ??
2. $\langle \rho(d^m) \rangle = 1/m$
3. Asymptotical PDF of filling the segment $[0, c(d^m)]$ for large $d^m$

$$\xi_m(s) = s^{m-1}/c(d^m), \quad \int \xi_3(s)ds = 1/m$$

LF (2006): Conjectures 2, 3 fail for $m = 3$:

$$4/9 < \langle \rho(d^3) \rangle < 1/2$$

Bourgain, Sinai (2007):
There exists the uniform limiting distribution for the normalized Frobenius number

$$X_3 = c(d^3) / \sqrt{\pi_3}$$

Ustinov (2008):

$$P(X_3) = \begin{cases} P_1(X_3) & \text{if} & 0 < X_3 < \sqrt{3}, \\ P_2(X_3) & \text{if} & \sqrt{3} < X_3 < 2, \\ P_3(X_3) & \text{if} & 2 < X_3 < \infty, \end{cases}$$
\[ P_1(X_3) = 0, \quad P_2(X_3) = \frac{12}{\pi} p_2(X_3), \quad P_3(X_3) = \frac{12}{\pi^2} p_3(X_3), \]

\[ p_2(X_3) = \frac{X_3}{\sqrt{3}} - \sqrt{4 - X^2} \]

\[ p_3(X_3) = X \sqrt{3} \arccos \frac{X_3 + 3 \sqrt{X_3^2 - 4}}{4 \sqrt{X_3^2 - 3}} + \frac{3}{2} \sqrt{X_3^2 - 4 \log \frac{X_3^2 - 4}{X_3^2 - 3}} \]

\[ P(X_3) = \frac{18}{\pi^2} \frac{1}{X_3^3} + O \left( \frac{1}{X_3^5} \right), \quad X_3 \to \infty, \]

\[ C_3 = \int_0^\infty X_3 P(X_3) dX_3 = \frac{8}{\pi}, \quad \int_0^\infty P(X_3) dX_3 = 1 \]

Marklof (2009), Strömbergsson (2011):

There exists the uniform limiting distribution for the normalized Frobenius number

\[ X_m = c(d^m) / m^{-1} \sqrt{\pi m} \]

\( P(X_m) \) has almost all of its mass concentrated in the interval

\[ m^{-1/2}(m-1)! < X_m < (1 + \eta) m^{-1/2}(m-1)!, \]

where \( \eta + e \log \eta = 0 \) and \( \eta \approx 0.756 \)

**Asymptotics of \( P(X_m) \)**

\[ P(X_m) = \binom{m}{2} \frac{X_m^{-m}}{\zeta(m-1)} + O \left( X_m^{-[m+1/2](m-2)]} \right), \quad X_m \to \infty \]

The functions \( P(X_m), 0 \leq X_m \leq \infty \), for \( m \geq 4 \) are still unknown!
Conjectures and questions

H. Wilf (1978)
\[ \rho(d^m) \geq \frac{1}{m} \]

\[ c(d^m) \leq 2\mu(d^m), \text{ Kaplan (2012)}, \quad c(d^m) \leq 3\mu(d^m), \text{ Eliahou (2016)} \]

M. Bras-Amores (2008)
The number \( N(g) \) of numerical semigroups of genus \( g \leq 15 \)

<table>
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<th>( g )</th>
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<td>1001</td>
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<td>2857</td>
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</table>

1) \( N(g) \geq N(g - 1) + N(g - 2), \quad g \geq 2 \)

2) \( \lim_{g \to \infty} \frac{N(g - 1) + N(g - 2)}{N(g)} = 1, \quad \text{Zhai (2013)} \)

3) \( \lim_{g \to \infty} \frac{N(g)}{N(g - 1)} = \phi, \quad \phi = \frac{1 + \sqrt{5}}{2}, \quad \text{Zhai (2013)} \)

LF (2016)

\[ F \left( \langle (n - 1)^{2q}, n^{2q}, (n + 1)^{2q} \rangle \right) = \mathcal{O} \left( n^{3q} \right) \]

\[ F \left( \langle (n - 1)^{2q+1}, n^{2q+1}, (n + 1)^{2q+1} \rangle \right) = \mathcal{O} \left( n^{3q+2} \right) \]

\( T_2 = 4, \quad T_3 = 18, \quad T_4 = 40, \quad T_{n \geq 5} = \ldots ? \)
what are the congruence moduli \( T_k \) for the higher \( k \)?
The problem comes up in many real-world contexts

Local Postage Stamp Problem

Having an unlimited supply of 17-cent, 18-cent, and 19-cent stamps we like to know what is the largest possible value of postage stamps on an envelope which we are unable to accomplish

Coin Change Problem

To determine the largest monetary amount that can not be obtained using only coins of specified denominations

Chicken McNugget Problem

Chicken McNuggets come in boxes of 6, 9, and 20 nuggets. To find the largest number of nuggets that could not have been bought.

References

- J. L. Ramírez-Alfonsín, The Diophantine Frobenius Problem, Oxford University Press, 2005
THANK YOU FOR ATTENTION