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Gabidulin Codes over Fields of Characteristic Zero and Their Applications

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Outline

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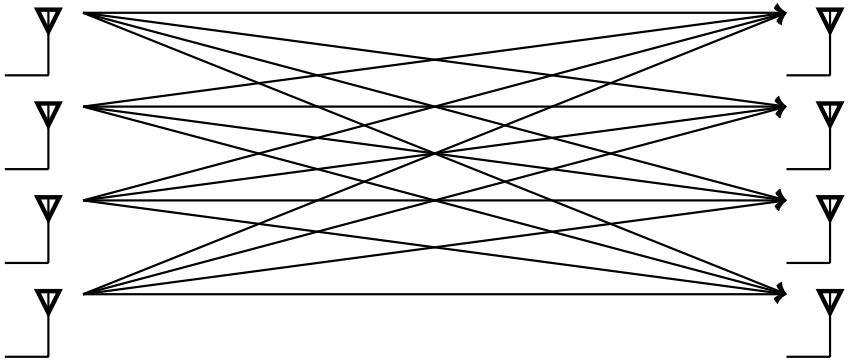

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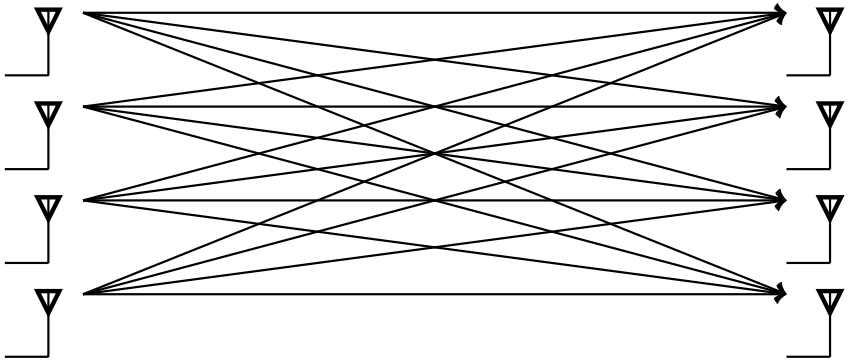

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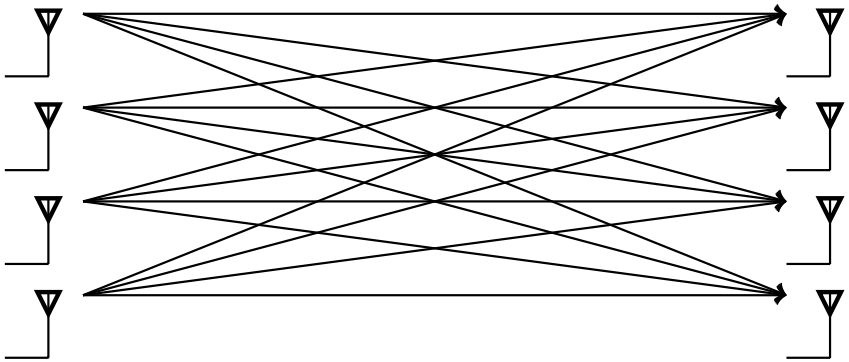

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- Space-Times Codes based on generalized Gabidulin codes (Robert [Rob15])

Compressed Sensing

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$$

- $\boldsymbol{b} \in \mathbb{C}^m$ (known)
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Hamming Metric Decoding Problem

$$\boldsymbol{s} = \boldsymbol{H}(\boldsymbol{c} + \boldsymbol{e}) = \boldsymbol{H}\boldsymbol{e}$$

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- $\boldsymbol{H} \in K^{n-k \times n}$ pc matrix (known)
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Low-Rank Matrix Recovery (LRMR)

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L/K Galois extension (i.e. normal and separable), $[L : K] =: m$

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Theorem *(Augot, Loidreau, Robert [ALR13])*

char_θ is square-free
$\Leftrightarrow L^\theta = \{x : \theta(x) = x\} = K$
$\Leftrightarrow \mathrm{Gal}\,(L/K)$ is cyclic and θ is a generator

Examples

General	Finite Fields	Cyclotomic Extensions
K	$\mathbb{F}_q$	$\mathbb{Q}$
L	$\mathbb{F}_{q^m}$	$\mathbb{Q}(\zeta_n)$, $\zeta_n = \mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}$ (n-th root of unity)
$[L:K]$	m	$\varphi(n)$
θ	$\cdot^q$ (Frobenius aut.)	i s.t. $\gcd(i,n)=1$ $\theta_i : \zeta_n^j \mapsto (\zeta_n^j)^i$

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Example: $n = 4 \Rightarrow \zeta_4 = \mathrm{i}$, $\mathbb{Q}(\zeta_4) = \mathbb{Q}(\mathrm{i}) = \{a + \mathrm{i}b : a, b \in \mathbb{Q}\}$, $\varphi(4) = 2$

$$\theta_1 : 1 \mapsto 1^1 = 1, \mathrm{i} \mapsto \mathrm{i}^1 = \mathrm{i} \quad \Rightarrow a + \mathrm{i}b \mapsto a + \mathrm{i}b \ (\mathrm{char}_\theta \textbf{ not square-free})$$

$$\theta_3 : 1 \mapsto 1^3 = 1, \mathrm{i} \mapsto \mathrm{i}^3 = -\mathrm{i} \quad \Rightarrow a + \mathrm{i}b \mapsto a - \mathrm{i}b \ (\mathrm{char}_\theta \text{ square-free})$$

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Other Field Extensions: Kummer and Artin-Schreier extensions

L/K field extension, $\theta \in \mathrm{Gal}\,(L/K)$.

$$L[x;\theta] = \left\{ a = \sum_{i=0}^{d} a_i x^i \;:\; a_i \in \mathbb{F}_{q^m}, d \in \mathbb{N} \right\}$$

θ-Polynomials

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Properties

- $(L[x;\theta], +, \cdot)$ is a ring
- $(L[x;\theta], +, \cdot)$ is non-commutative in general
- θ-polynomials are skew polynomials (without derivation)
- Isomorphic to linearized polynomials in case $\mathbb{F}_{q^m}/\mathbb{F}_q$, $\theta = \cdot^q$

θ-Polynomials (cont'd)

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- $(a \cdot b)(\alpha) = a(b(\alpha))$
- $\deg(a \cdot b) = \deg a + \deg b$
- Zeros of a are subspace[1] of L
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If char_θ is square-free,

$$\dim \ker(a) \leq \deg a$$

Annihilator Polynomial

Subspace $\mathcal{U} \subseteq L \Rightarrow \exists$ monic $\mathcal{A}_{\mathcal{U}} \in L[x;\theta]$ of minimal degree:

$$\mathcal{A}_{\mathcal{U}}(u) = 0 \quad \forall u \in \mathcal{U}$$

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$x_1, \ldots, x_\ell \in L$, linearly independent over K. $y_1, \ldots, y_\ell \in L$ arbitrary.
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$$\deg \mathcal{A}_\mathcal{U} = \dim \mathcal{U} \quad \text{(in general: } \deg \mathcal{A}_\mathcal{U} \leq \dim \mathcal{U}\text{)}$$
$$\deg \mathcal{I} < \ell \quad \text{(and } \mathcal{I} \text{ is unique)}$$

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L/K field extension, $\theta \in \mathrm{Gal}\,(L/K)$ with char_θ square-free.

Definition

$g_1, \ldots, g_n \in L$, linearly independent over K, $k \leq n \leq m = [L : K]$

$$\mathcal{C}_{\mathrm{G}}[n,k] = \left\{ \boldsymbol{c} = \left[f(g_1), \ldots, f(g_n) \right] : f \in L[x;\theta] \wedge \deg f < k \right\} \subseteq L^n$$

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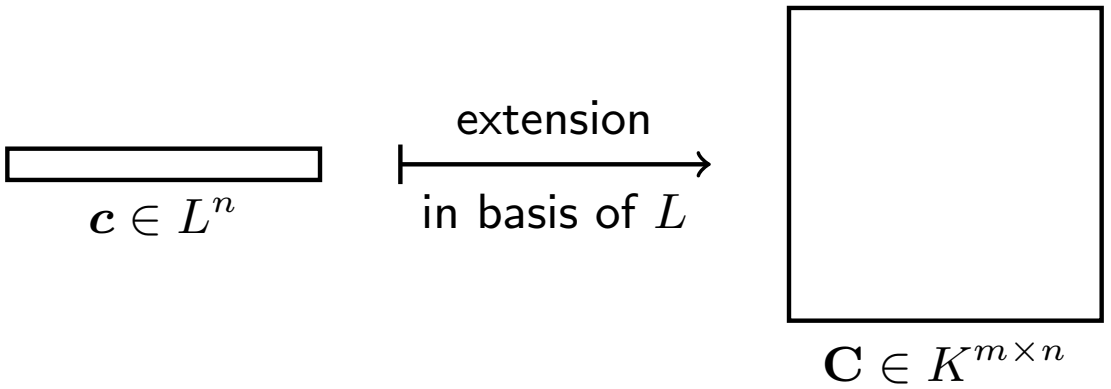

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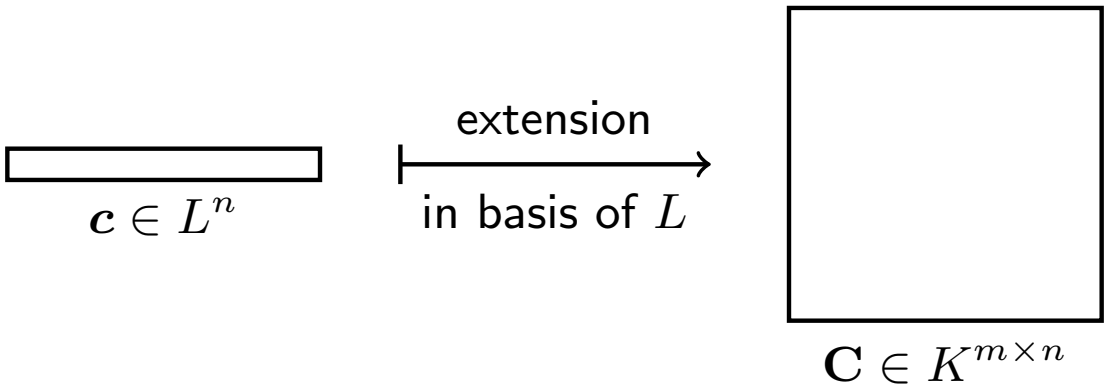

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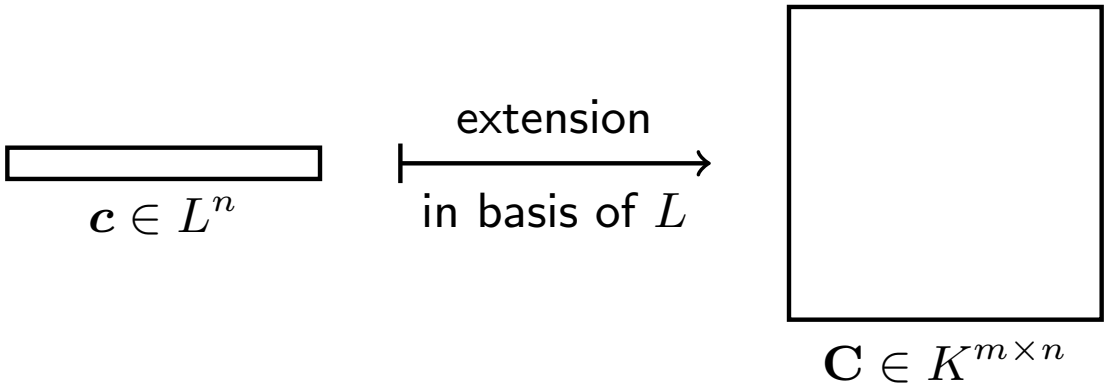

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Minimum rank distance $d = \min_{\boldsymbol{c}_1 \neq \boldsymbol{c}_2} \mathrm{d}_{\mathrm{R}}(\boldsymbol{c}_1, \boldsymbol{c}_2) = n - k + 1$ (MRD)

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(Also true for char_θ not square-free)

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- Similar to the Extended Euclidean Algorithm (EEA)
- Advantage: Coefficient size reduction in intermediate computations

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Algorithm: Decode Gabidulin Codes

Input: $\boldsymbol{r} = \boldsymbol{c} + \boldsymbol{e}$

Output: f s.t. $\boldsymbol{c} = [f(g_1), \ldots, f(g_n)]$ or "decoding failure".

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4 $(\chi, \varrho) \leftarrow$ Right-divide Ω by Λ

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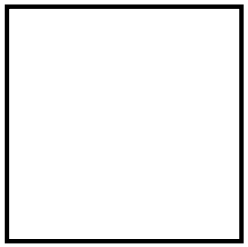
- If $\mathrm{wt_R}(\boldsymbol{e}) < \frac{d}{2}$, the algorithm finds f
- Complexity/coefficient size growth tradeoff:

	Fast	**"Normal"**	**Small growth**
Operations in L	$O(n^{1.69})$	$O(n^2)$	$O(n^3)$
Row Reduction	EEA [Wac13] or D&Q [PMM+16]	[PNLS15]	Fraction-free [BCL06]

Back to LRMR (K-linear map $\mathcal{A}$)



$\boldsymbol{E} \in K^{m \times n}$

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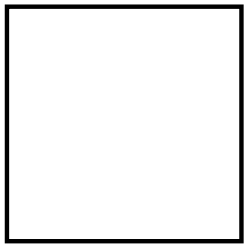

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Inverse extension in basis of L

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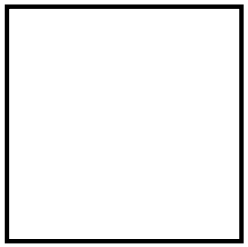

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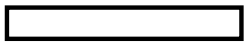

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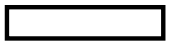
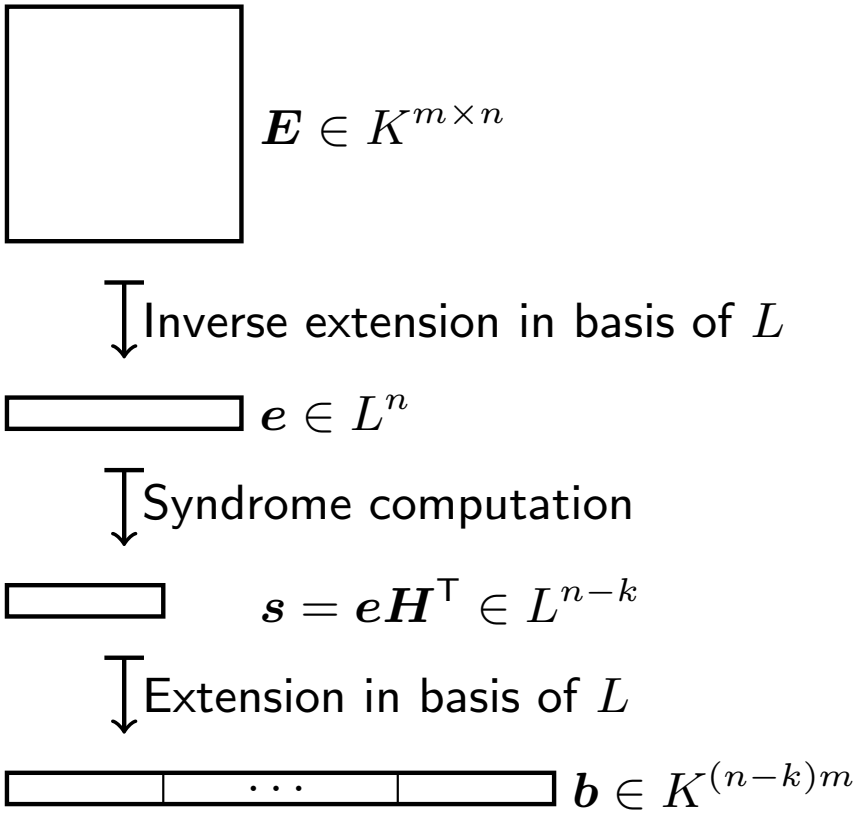
Syndrome computation



$\boldsymbol{s} = \boldsymbol{e}\boldsymbol{H}^{\mathsf{T}} \in L^{n-k}$

Back to LRMR (K-linear map $\mathcal{A}$)



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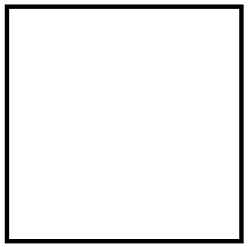

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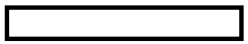
(linear) Inverse extension in basis of L



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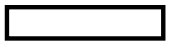

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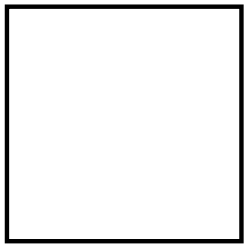

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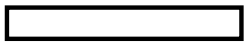

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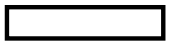

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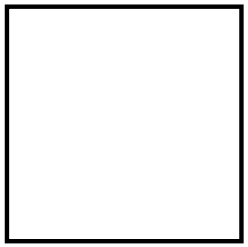

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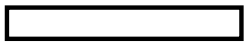

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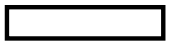

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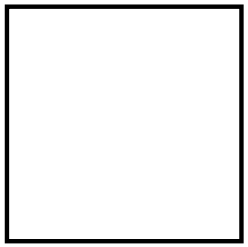

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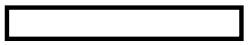

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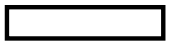

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(linear) Extension in basis of L

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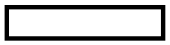
Find a solution of $\boldsymbol{r}\boldsymbol{H}^\mathsf{T} = \boldsymbol{s}$



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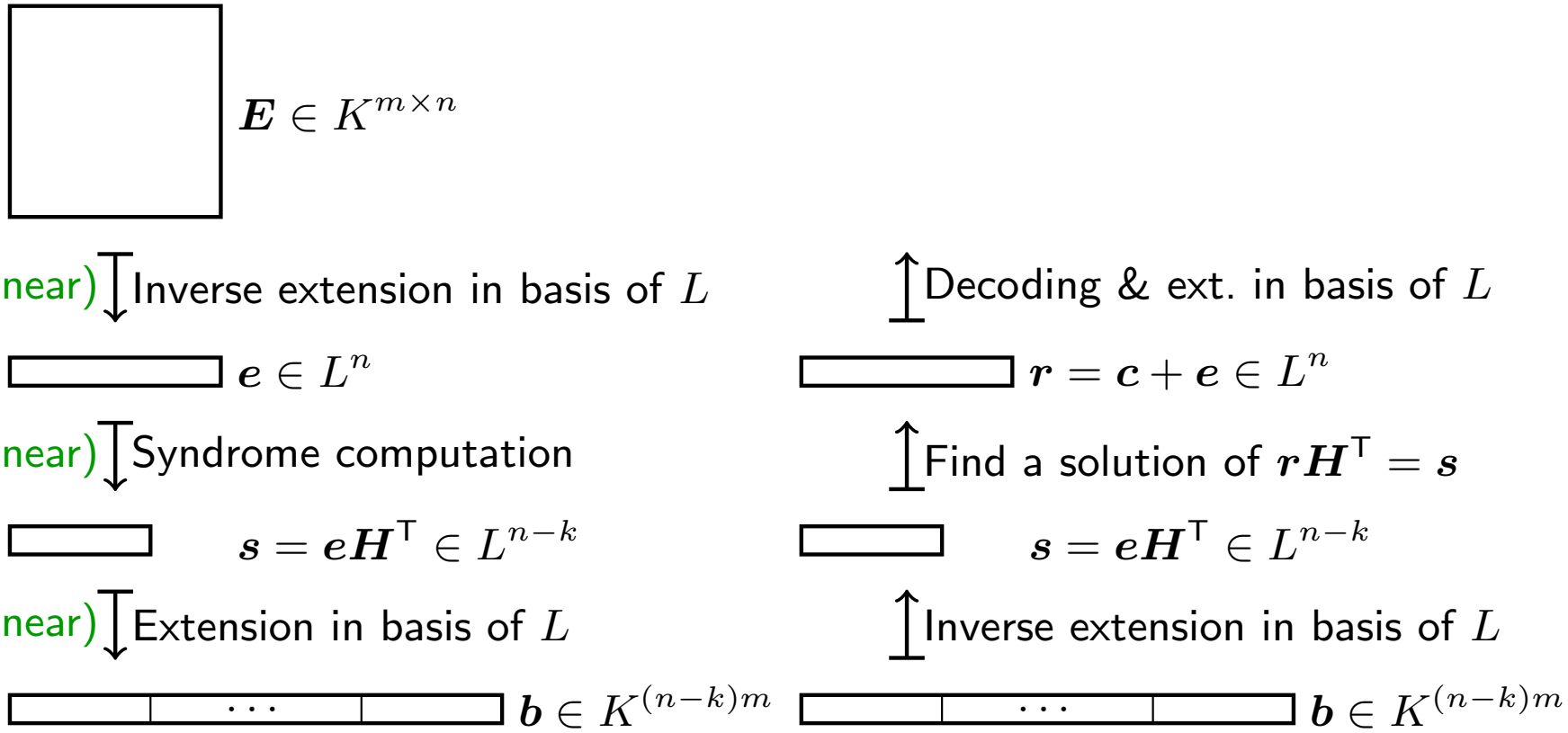

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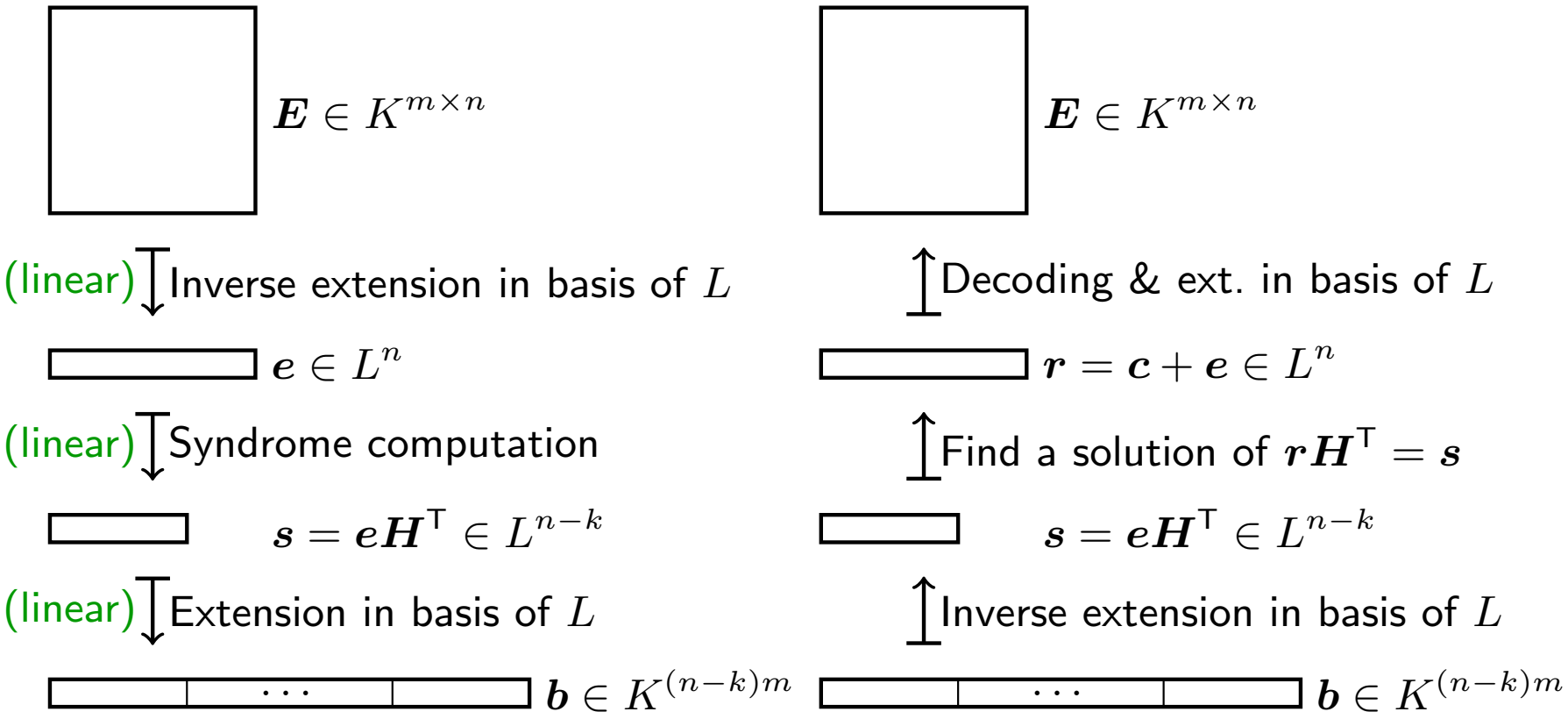

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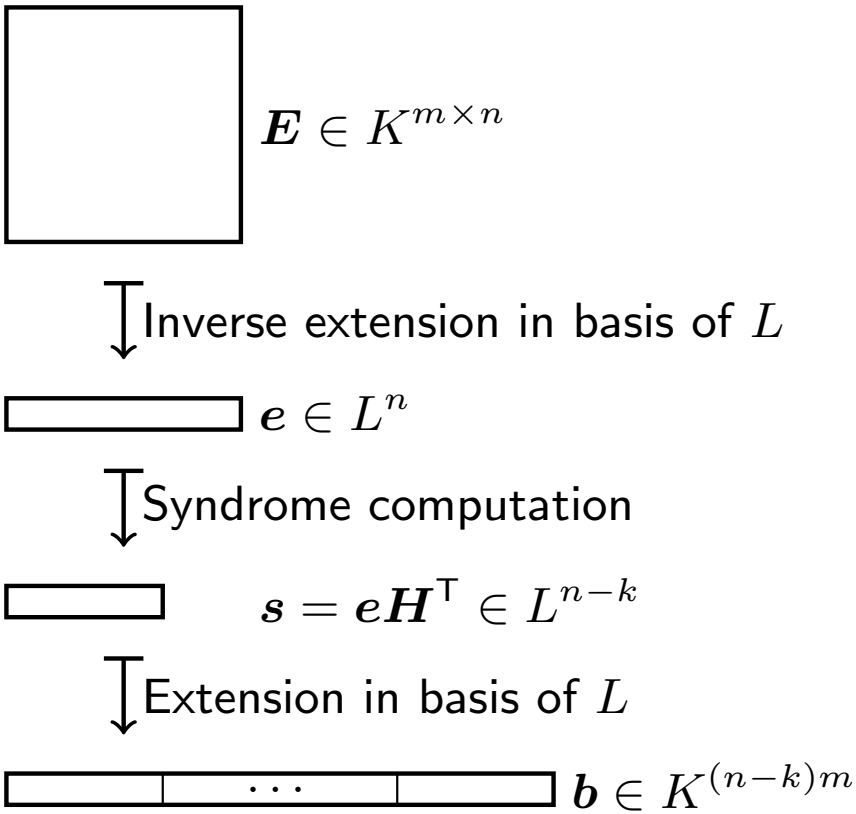
Needed: $K \in \{\mathbb{R}, \mathbb{C}\}$. Possible L:



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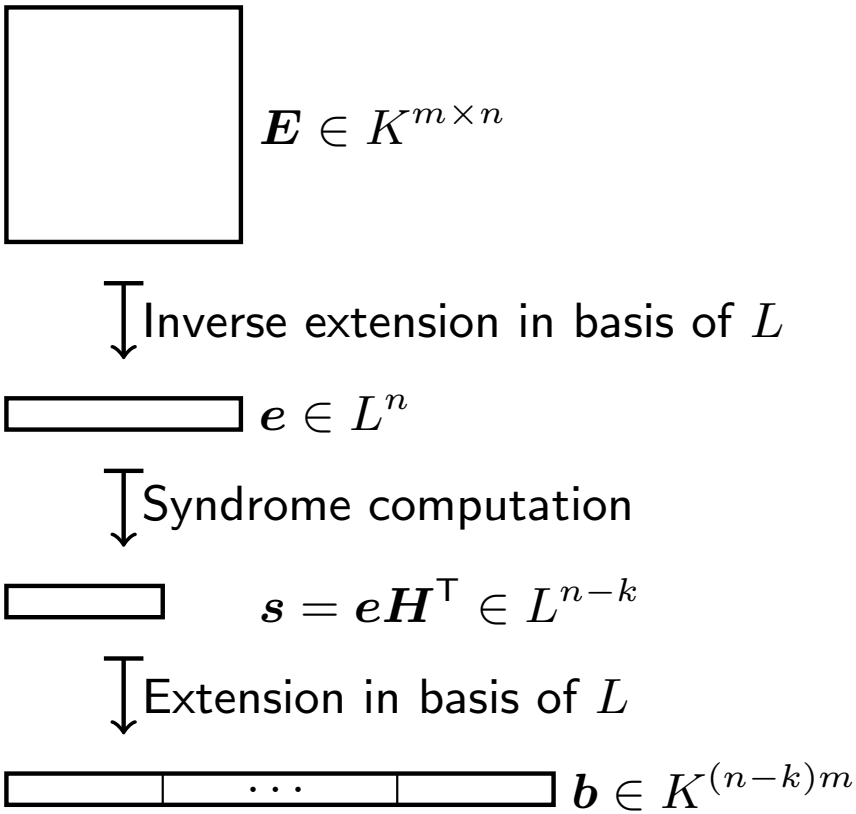

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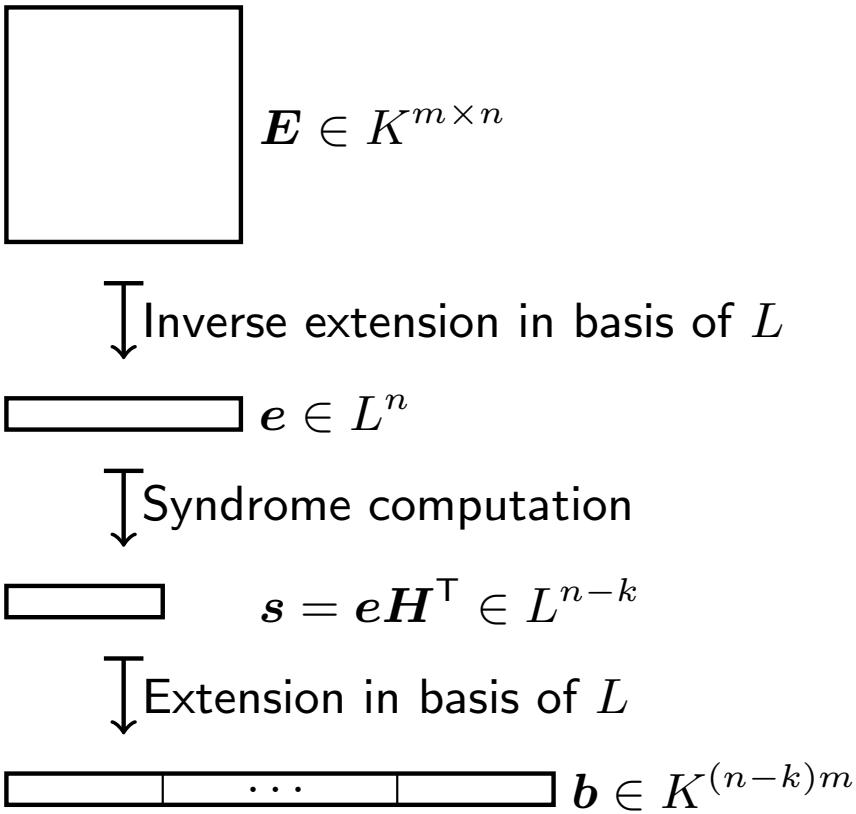

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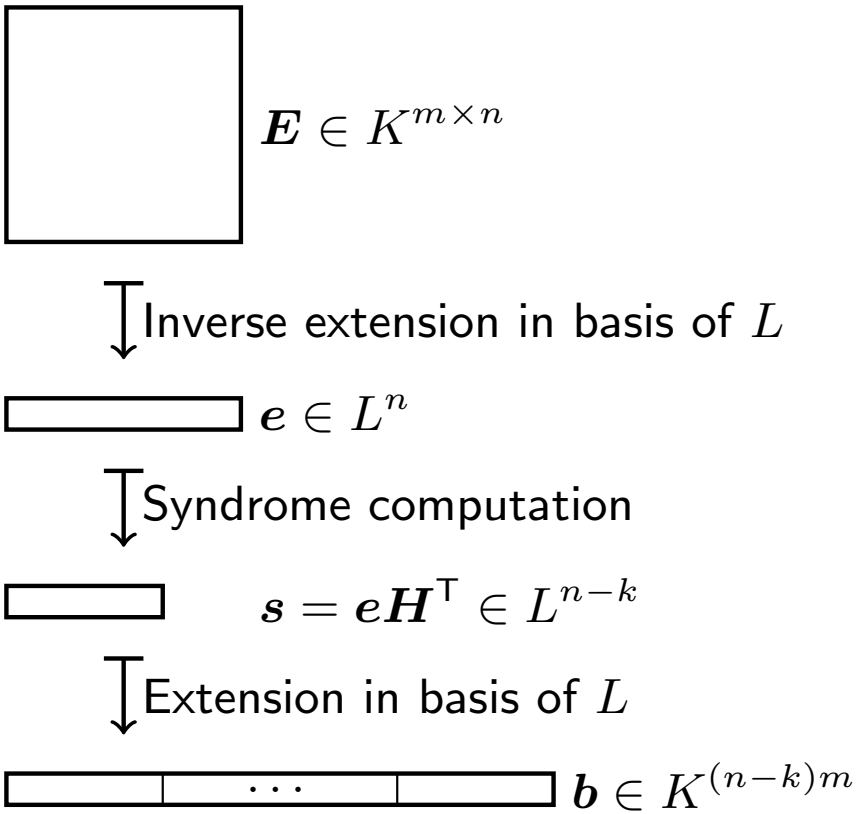

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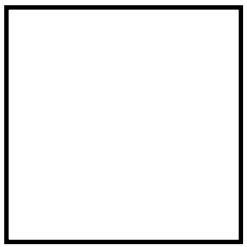
Idea: Choose K to be a **dense** subfield of $\mathbb{R}$ or $\mathbb{C}$, e.g.,

$K \subseteq$	$\mathbb{R}$	$\mathbb{C}$
K	$\mathbb{Q}$	$\mathbb{Q}(\zeta_r)$
L	$\mathbb{Q}(\zeta_r)$	Kummer extension
m	$\varphi(r)$	r

Back to LRMR (Choice of K and L)

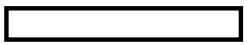
$\boldsymbol{X} \in \mathbb{C}^{m \times n}$

Rank-preserving mapping



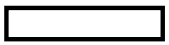
$\boldsymbol{E} \in K^{m \times n}$

Inverse extension in basis of L



$\boldsymbol{e} \in L^n$

Syndrome computation



$\boldsymbol{s} = \boldsymbol{e}\boldsymbol{H}^{\mathsf{T}} \in L^{n-k}$

Extension in basis of L

$\boldsymbol{b} \in K^{(n-k)m}$

Needed: $K \in \{\mathbb{R}, \mathbb{C}\}$. Possible L:

- $K = \mathbb{R}$: $L \in \{\mathbb{R}, \mathbb{C}\}$ $(m \leq 2)$
- $L = \mathbb{C}$: $L = \mathbb{C}$ $(m = 1)$

Idea: Choose K to be a **dense** subfield of $\mathbb{R}$ or $\mathbb{C}$, e.g.,

$K \subseteq$	$\mathbb{R}$	$\mathbb{C}$
K	$\mathbb{Q}$	$\mathbb{Q}(\zeta_r)$
L	$\mathbb{Q}(\zeta_r)$	Kummer extension
m	$\varphi(r)$	r

Outline

If we replace

$$\begin{aligned} \mathbb{F}_{q^m}/\mathbb{F}_q &\longleftrightarrow L/K \\ \cdot^q &\longleftrightarrow \theta \in \mathrm{Gal}\left(L/K\right), \mathrm{char}_\theta \text{ square-free,} \end{aligned}$$

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then the

- Definition of Rank Metric
- Definition of Gabidulin codes
- Decoding of Gabidulin codes

immediately generalize.

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Applications

- Space–Time codes
- Low-Rank Matrix Recovery

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