Gabidulin Codes over Fields of Characteristic Zero and Their Applications

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Outline

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1 Motivation

2 Preliminaries

3 Gabidulin Codes

4 Conclusion

History

1978/1985/1991 Gabidulin codes over finite fields

Delsarte [Del78], Gabidulin [Gab85], Roth [Rot91]

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2016	Application: Low-Rank Matrix Recovery Müelich, Puchinger, Bossert [MPB16]

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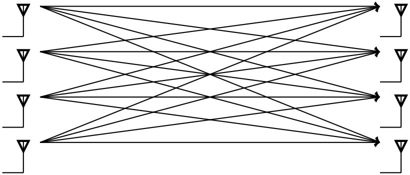

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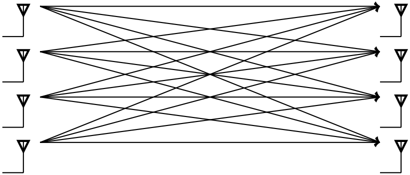

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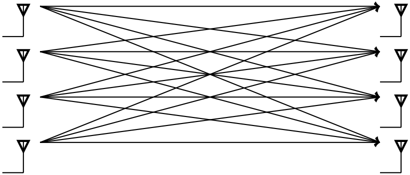

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- Space-Times Codes based on generalized Gabidulin codes (Robert [Rob15])

Low-Rank Matrix Recovery (LRMR)

Compressed Sensing

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$$

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Theorem *(Augot, Loidreau, Robert [ALR13])*

char_θ is square-free
$\Leftrightarrow L^\theta = \{x : \theta(x) = x\} = K$
$\Leftrightarrow \mathrm{Gal}\,(L/K)$ is cyclic and θ is a generator

General	Finite Fields	Cyclotomic Extensions
K	$\mathbb{F}_q$	$\mathbb{Q}$
L	$\mathbb{F}_{q^m}$	$\mathbb{Q}(\zeta_n)$, $\zeta_n = \mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}$ (n-th root of unity)
$[L:K]$	m	$\varphi(n)$
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Example: $n = 4 \Rightarrow \zeta_4 = \mathrm{i}$, $\mathbb{Q}(\zeta_4) = \mathbb{Q}(\mathrm{i}) = \{a + \mathrm{i}b : a, b \in \mathbb{Q}\}$, $\varphi(4) = 2$

$\theta_1 : 1 \mapsto 1^1 = 1, \mathrm{i} \mapsto \mathrm{i}^1 = \mathrm{i} \quad \Rightarrow a + \mathrm{i}b \mapsto a + \mathrm{i}b$ (char_θ **not** square-free)

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Other Field Extensions: Kummer and Artin-Schreier extensions

L/K field extension, $\theta \in \mathrm{Gal}\,(L/K)$.

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Properties

- $(L[x;\theta], +, \cdot)$ is a ring
- $(L[x;\theta], +, \cdot)$ is non-commutative in general
- θ-polynomials are skew polynomials (without derivation)
- Isomorphic to linearized polynomials in case $\mathbb{F}_{q^m}/\mathbb{F}_q$, $\theta = \cdot^q$

θ-Polynomials (cont'd)

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- $\deg(a \cdot b) = \deg a + \deg b$
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- $a(\cdot) : L \to L$ is an K-linear map

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If char_θ is square-free,

$$\dim \ker(a) \leq \deg a$$

Annihilator Polynomial

Subspace $\mathcal{U} \subseteq L \Rightarrow \exists$ monic $\mathcal{A}_{\mathcal{U}} \in L[x;\theta]$ of minimal degree:

$$\mathcal{A}_{\mathcal{U}}(u) = 0 \quad \forall u \in \mathcal{U}$$

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$x_1, \dots, x_\ell \in L$, linearly independent over K. $y_1, \dots, y_\ell \in L$ arbitrary.
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$$\begin{aligned} \deg \mathcal{A}_{\mathcal{U}} &= \dim \mathcal{U} && \text{(in general: } \deg \mathcal{A}_{\mathcal{U}} \leq \dim \mathcal{U}\text{)} \\ \deg \mathcal{I} &< \ell && \text{(and } \mathcal{I} \text{ is unique)} \end{aligned}$$

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L/K field extension, $\theta \in \mathrm{Gal}\,(L/K)$ with char_θ square-free.

Definition

$g_1, \ldots, g_n \in L$, linearly independent over K, $k \leq n \leq m = [L : K]$

$$\mathcal{C}_{\mathrm{G}}[n,k] = \left\{ \boldsymbol{c} = \big[f(g_1), \ldots, f(g_n) \big] : f \in L[x;\theta] \wedge \deg f < k \right\} \subseteq L^n$$

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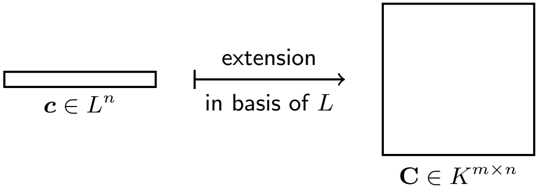

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$$\boldsymbol{c} \in L^n \quad \xmapsto[\text{in basis of } L]{\text{extension}} \quad \mathbf{C} \in K^{m \times n}$$

Rank Metric[1]: $\quad \mathrm{wt}_{\mathrm{R}}(\boldsymbol{c}) = \operatorname{rank}(\mathbf{C}), \quad \mathrm{d}_{\mathrm{R}}(\boldsymbol{c}_1, \boldsymbol{c}_2) = \operatorname{rank}(\mathbf{C}_1 - \mathbf{C}_2)$

[1] Several definitions of rank metric exist. They are equivalent if $\operatorname{char}_\theta$ is square-free, cf. [ALR13]

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Minimum rank distance $d = \min_{\boldsymbol{c}_1 \neq \boldsymbol{c}_2} \mathrm{d}_{\mathrm{R}}(\boldsymbol{c}_1, \boldsymbol{c}_2) = n - k + 1$ (MRD)

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- $\Lambda := \mathcal{A}_{\langle e_1, \dots, e_n \rangle}$ (unknown error span polynomial)
- $\hat{r}$ interpolation polynomial with $\hat{r}(g_i) = r_i \ \forall i$ (known)

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$$\Lambda \hat{r} \equiv \Lambda f \mod \mathcal{A}_{\langle g_1, \dots, g_n \rangle}$$

*characteristic zero equivalent to Gao's key equation for finite field Gabidulin codes (Wachter-Zeh [Wac13])

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(Also true for char_θ not square-free)

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Linear Shift Register (LSR) Synthesis Problem

Given $\hat{r}, \mathcal{A}_{\langle g_1, \dots, g_n \rangle}$, find non-zero $(\lambda, \omega) \in L[x;\theta]^2$ with $\deg \lambda$ minimal and

$$\lambda \hat{r} \equiv \omega \mod \mathcal{A}_{\langle g_1, \dots, g_n \rangle}$$

$$\deg \lambda + k > \deg \omega$$

Key Equation (Müelich, Puchinger, Mödinger, Bossert [MPMB16])

$$\Lambda\hat{r} \equiv \Lambda f \mod \mathcal{A}_{\langle g_1,\dots,g_n\rangle}$$

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Assumption that char_θ is square-free is necessary!

Solving LSR Using Row Reduction

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Row reduction ($\boxed{\cdot}$ is leading position = rightmost pos. of max. degree in row)

$$\begin{bmatrix} x^k & \boxed{\hat{r}} \\ 0 & \boxed{\mathcal{A}_{\langle g_1,\dots,g_n\rangle}} \end{bmatrix}$$

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- Similar to the Extended Euclidean Algorithm (EEA)
- Advantage: Coefficient size reduction in intermediate computations

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Decoding Summary

Algorithm: Decode Gabidulin Codes

Input: $\boldsymbol{r} = \boldsymbol{c} + \boldsymbol{e}$
Output: f s.t. $\boldsymbol{c} = [f(g_1), \ldots, f(g_n)]$ or "decoding failure".

```
Calculate r̂ and A_⟨g_1,...,g_n⟩
(λ, ω) ← Solve LSR with input r̂, A_⟨g_1,...,g_n⟩ using row reduction
(Λ, Ω) ← α^{-1}(λ, ω)
(χ, ϱ) ← Right-divide Ω by Λ
if ϱ = 0 then return χ
else return "decoding failure"
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Calculate $\hat{r}$ and $\mathcal{A}_{\langle g_1,\ldots,g_n\rangle}$
$(\lambda, \omega) \leftarrow$ Solve LSR with input $\hat{r}$, $\mathcal{A}_{\langle g_1,\ldots,g_n\rangle}$ using row reduction
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- If $\mathrm{wt}_{\mathrm{R}}(\boldsymbol{e}) < \frac{d}{2}$, the algorithm finds f

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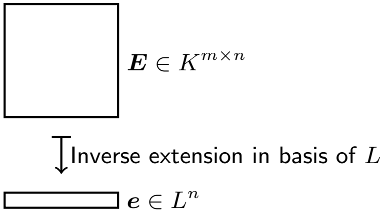
- If $\mathrm{wt}_{\mathrm{R}}(\boldsymbol{e}) < \frac{d}{2}$, the algorithm finds f
- Complexity/coefficient size growth tradeoff:

	Fast	**"Normal"**	**Small growth**
Operations in L	$O(n^{1.69})$	$O(n^2)$	$O(n^3)$
Row Reduction	EEA [Wac13] or D&Q [PMM+16]	[PNLS15]	Fraction-free [BCL06]

Back to LRMR (K-linear map $\mathcal{A}$)

$\boldsymbol{E} \in K^{m \times n}$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Inverse extension in basis of L
$\boldsymbol{e} \in L^n$

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$\downarrow$ Inverse extension in basis of L

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$\downarrow$ Syndrome computation

$\boldsymbol{s} = \boldsymbol{e}\boldsymbol{H}^\mathsf{T} \in L^{n-k}$

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$\downarrow$ Extension in basis of L

$\boldsymbol{b} \in K^{(n-k)m}$

$\boldsymbol{E} \in K^{m \times n}$

(linear) $\downarrow$ Inverse extension in basis of L

$\boldsymbol{e} \in L^n$

(linear) $\downarrow$ Syndrome computation

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Theorem *Müelich, Puchinger, Bossert [MPB16]*

If $\operatorname{rank}(\boldsymbol{E}) \leq \frac{d-1}{2} = \frac{n-k}{2}$, $\boldsymbol{E}$ can be reconstructed from $\boldsymbol{b} = \mathcal{A}(\boldsymbol{E})$.

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(linear) Extension in basis of L | Inverse extension in basis of L

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Back to LRMR (Choice of K and L)

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Idea: Choose K to be a **dense** subfield of $\mathbb{R}$ or $\mathbb{C}$, e.g.,

$K \subseteq$	$\mathbb{R}$	$\mathbb{C}$
K	$\mathbb{Q}$	$\mathbb{Q}(\zeta_r)$
L	$\mathbb{Q}(\zeta_r)$	Kummer extension
m	$\varphi(r)$	r

Back to LRMR (Choice of K and L)

$\boldsymbol{X} \in \mathbb{C}^{m \times n}$

$\downarrow$ Rank-preserving mapping

$\boldsymbol{E} \in K^{m \times n}$

$\downarrow$ Inverse extension in basis of L

$\boldsymbol{e} \in L^{n}$

$\downarrow$ Syndrome computation

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Outline

Conclusion

If we replace

$$\begin{array}{rcl} \mathbb{F}_{q^m}/\mathbb{F}_q & \longleftrightarrow & L/K \\ \cdot^{q} & \longleftrightarrow & \theta \in \mathrm{Gal}\left(L/K\right), \ \mathrm{char}_\theta \text{ square-free,} \end{array}$$

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- Definition of Rank Metric
- Definition of Gabidulin codes
- Decoding of Gabidulin codes

immediately generalize.

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Applications

- Space–Time codes
- Low-Rank Matrix Recovery

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